

TRAJECTORY BASED MODELS. EVALUATION OF MINMAX PRICING BOUNDS.

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ABSTRACT. The paper studies market models based on trajectory spaces, properties of such models are obtained without recourse to probabilistic assumptions. For a given European option, an interval of rational prices exists under a more general condition than the usual no-arbitrage requirement. The paper develops computational results in order to evaluate the option bounds; the global minmax optimization, defining the price interval, is reduced to a local minmax optimization via dynamic programming. A general class of trajectory sets is described for which the market model introduced by Britten Jones and Neuberger [7] is nested as a particular case. We also develop a market model based on an operational setting constraining market movements and investor's portfolio rebalances. Numerical examples are presented, the effect of the presence of arbitrage on the price bounds is illustrated.

1. INTRODUCTION

Stock chart values are an observable quantity and, as such, it is relevant to obtain market properties based solely on their characteristics. Results obtained in that way, will be valid independently of the probability distribution imposed on the trajectory set modelling the stock charts. Notice that financial market models based on stochastic processes implicitly incorporate a set of trajectories; namely, the support of the process. The latter is a by-product of probabilistic assumptions while our proposed approach considers trajectory sets as a primary modelling object. Theoretical developments in [9] (following [7]) show that, in a general discrete setting, there are natural conditions on the trajectory set which provide a worst case price interval for general European options. The results are probability free and the framework allows for a weakening of the no-arbitrage requirement. The basic results follow from trajectory assumptions guaranteeing that the option with zero payoff has zero price. Natural connections with the well established stochastic approach are also described in [9].

The present paper provides results that allow to evaluate the worst case price interval, originally given by a global minmax optimization, in terms of an iterative, dynamic programming, based optimization. We also show by example, how one can construct trajectory spaces by means of observable functionals defined on stock charts as well as other observable sources of uncertainty. In particular, we put forward specific trajectory based models of a single tradable financial underlying (a zero interest rate bank account is also assumed.)

The general class of models included in the formalism allow for certain arbitrage opportunities while, at the same time, providing a coherent price interval for options without introducing logical or practical inconsistencies. Several important properties of the proposed models are proven and

practical computational approaches are also described with some detail. Relevant numerical examples illustrate the viability of the approach and some of the characteristics of the models studied.

The general framework is a discrete market model $\mathcal{M} = \mathcal{S}^{\mathcal{W}} \times \mathcal{H}$ where $\mathbf{S} \in \mathcal{S}^{\mathcal{W}}$ are the trajectories consisting of a sequence of pairs $\{(S_i, W_i)_{i \geq 0}\}$. The real numbers S_i represent possible stock chart values and W_i encodes other, observable, sources of uncertainty linked to the unfolding of the stock. The framework allows W_i to take values in an arbitrary set. Investment portfolios $H = \{H_i\}_{i \geq 0}$ are given by functions $H_i : \mathcal{S}^{\mathcal{W}} \rightarrow \mathbb{R}$ that are non-anticipative and self-financing when considered together with the bank account. Notice that we work in a discrete setting in the sense that portfolio rebalances are indexed by integers but the data $x(t)$ (stock charts) can be considered to flow in a continuous time manner. On the other hand, financial transactions are naturally discrete and hence we obtain samples $S_i = x(t_i)$.

Given an European option, with an appropriate payoff function Z and under basic conditions on $\mathcal{S}^{\mathcal{W}}$, there exists a price interval $[\underline{V}(S_0, Z, \mathcal{M}), \bar{V}(S_0, Z, \mathcal{M})]$ (S_0 is the initial stock value, common to all trajectories). These bounds are trajectory based and, as such, need to be evaluated by a minmax optimization. This framework has been fully developed in [9] which, in turn, built on the specific model introduced in [7] (see also the book exposition in [14]). The approach can be considered a non-probabilistic financial model which uses a trajectory space as a central object for modelling. The approach in continuous time is discussed in [1] and [2]. There is a somewhat disjointed literature that approaches financial modelling in a similar way to what we do in our paper, we mention, as an example, [4].

We present general and rigorous computational results that allow to evaluate the global minmax optimization, defining the price interval, by means of an iterative algorithm. To efficiently deal with the local optimization, a geometric algorithm, the so called *convex hull*, was introduced informally in [7] and used to reduce the local minmax to a single maximization. This last step is achieved by parametrizing the hedging parameter by a geometric ratio and represents the essence of the convex hull algorithm. The hedging ratio is a discretized version of the delta hedging term appearing in the stochastic setting. We provide a formal analysis of the procedure while extending it to a general setting. If the payoff function is convex the maximization coincides with the result in [8].

A general example of a trajectory set was introduced and explained with some detail in [9], specific instances of that model allow for trajectory dependent values of quadratic variation. The paper develops further this class of examples, in particular, values of W_i are proposed and related to observable characteristics of an observable chart $x(t)$ up to time t_i . Examples for W_i could be quadratic variation, time, business time, etc., the latter interpreted as the volume of transactions along $x(t)$ up to time t_i . A particular case, dealing with sampled quadratic variation, is developed with some detail in Section 6.2. Necessary conditions satisfied by chart samples are obtained based on how investors interact with chart data in an operational way. The trajectory examples $\mathcal{S}^{\mathcal{W}}$ presented in the paper substantially generalize the basic example introduced in [7]. In particular, our trajectory sets allow the presence of 0-neutral nodes which are arbitrage nodes.

The paper is organized as follows, Section 2 provides the general framework of the paper and describes notation and relevant results from [9] to be used in the remaining of the paper. That section introduces the notion of 0-neutral and its relationship with no-arbitrage. Section 2.2 indicates how to generalize all previous results to the case when trajectories are augmented to allow for another

source of uncertainty denoted by W . This is a crucial extension for the remaining of the paper. Section 3 describes a main result from [9] establishing the existence of a pricing interval under 0-neutrality hypothesis. We also present conditions on the option's payoffs guaranteeing the existence of the price bounds. Section 4 establishes, under appropriate conditions, how to recover the global minmax optimization defining the bound prices by means of an iterative dynamic programming procedure. Section 5 describes how the iterative procedure described in Section 4 can actually be implemented by an efficient, geometrically based, algorithm named *convex hull*. This algorithm was introduced in [7] for a specific model, we have formalized the result and extended it to our general setting. Section 6 describes a general class of models and a particular case given by an operational setup involving sampled quadratic variation. Section 7 describes a data structure supporting the implementation of the models introduced. Section 8 presents numerical experiments and discuss their financial meaning. Section 9 concludes by providing a perspective of the paper as well as some speculation on possible extensions. Appendix A provides most of the technical details, as well as some main statements, for the material presented in Section 3 related to conditions on option payoffs. Appendix B contain technical results related to Section 4. Appendix C presents the convex hull algorithm for nodes which are strictly 0-neutral. Finally Appendix D gives some auxiliary results.

2. GENERAL FRAMEWORK

We present briefly some of the basic notions we will need from [9] and refer to that reference for details.

Definition 1 (Trajectory and Portfolio Sets). *Given a real number s_0 a set of (discrete) trajectories $\mathcal{S} = \mathcal{S}(s_0, \Sigma)$ is a subset of the following set*

$$\mathcal{S}_\infty = \mathcal{S}_\infty(s_0) = \{S = \{S_i\}_{i \geq 0} : S_i \in \Sigma_i, S_0 = s_0\},$$

where $\Sigma = \{\Sigma_i\}_{i \geq 0}$ is a family of fixed subsets of \mathbb{R} .

A portfolio H is a sequence of (pairs of) functions $H = \{\Phi_i = (B_i, H_i)\}_{i \geq 0}$ with $B_i, H_i : \mathcal{S} \rightarrow \mathbb{R}$. A portfolio H is said to be self-financing at $S \in \mathcal{S}$ if for all $i \geq 0$,

$$(2.1) \quad H_i(S) S_{i+1} + B_i(S) = H_{i+1}(S) S_{i+1} + B_{i+1}(S).$$

A portfolio H is called non-anticipative if for all $S, S' \in \mathcal{S}$, satisfying $S'_k = S_k$ for all $0 \leq k \leq i$, it then follows that $\Phi_i(S) = \Phi_i(S')$.

A trajectory based discrete market (or trajectory market for short) is defined by $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ where elements $H \in \mathcal{H}$ are non-anticipative and self-financing at each $S \in \mathcal{S}$. Moreover, for each $H \in \mathcal{H}$ there exist an integer $N = N_H(S)$ satisfying $H_k(S) = H_N(S) = 0$ for all $k \geq N = N_H(S)$ (this fact will be referred to as H being liquidated at N_H). The zero portfolio is assumed to belong to \mathcal{H} . We introduce a more general setting that allows for other sources of uncertainty in Section 2.2.

The following notion will be needed in several instances later in the paper.

Definition 2 (Bounded Discrete Market). *The market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ will be called bounded, if for each $H \in \mathcal{H}$ there exists $n = n(H)$ such that $N_H(S) \leq n$, for all $S \in \mathcal{S}$.*

Given $(S, H) \in \mathcal{M}$, the self-financing property (2.1) implies that the portfolio value, defined by $V_H(i, S) = B_i(S) + H_i(S) S_i$ equals:

$$V_H(i, S) = V_H(0, S_0) + \sum_{k=0}^{i-1} H_k(S) (S_{k+1} - S_k).$$

Of course, $V_H(0, S_0) \equiv V_H(0, S) = B_0(S) + H_0(S) S_0$.

Clearly, to specify self-financing portfolios, it is enough to provide sequences $H = \{H_i\}$ of non-anticipative functions and an associated real number $V_0 = V_H(0, S_0)$.

For some results we will need the following definition.

Definition 3 (Stopping Time). *Given a trajectory space \mathcal{S} a trajectory based stopping time (or stopping time for short) is a function $\nu : \mathcal{S} \rightarrow \mathbb{N}$ such that if $S, S' \in \mathcal{S}$ and $S_k = S'_k$ for $0 \leq k \leq \nu(S)$ then $\nu(S') = \nu(S)$.*

For $S \in \mathcal{S}$ we will use the notation $\Delta_i S \equiv S_{i+1} - S_i$ for $i \geq 0$. Whenever convenient, the tuple (S, k) or the triple (S, H, k) will be referred generically as *a node*.

The following conditional spaces will play a key role. Given \mathcal{M} , for $S \in \mathcal{S}$ and $k \geq 0$ set:

$$\mathcal{S}_{(S,k)} \equiv \{\tilde{S} \in \mathcal{S} : \tilde{S}_i = S_i, 0 \leq i \leq k\}.$$

Notice $\mathcal{S}_{(S,0)} = \mathcal{S}$ and that if $S' \in \mathcal{S}_{(S,k)}$, then $\mathcal{S}_{(S',k)} = \mathcal{S}_{(S,k)}$. Consequently, the family $\{\mathcal{S}_{(S,k)} : S \in \mathcal{S}\}$ is a partition of \mathcal{S} , which will be useful for some results as in Theorems 3 and 4.

The framework introduced can be used to model continuous time market variables which are observed in a discrete fashion. So, the numbers S_i can be considered as event driven samples from continuous data $x(t)$. A more detailed discussion is presented in [9].

2.1. Global, Conditional and Local Concepts. We use the following definition of no-arbitrage market.

Definition 4 (Arbitrage-Free Market). *Given a discrete market \mathcal{M} , we will call $H \in \mathcal{H}$ an arbitrage strategy if:*

- $\forall S \in \mathcal{S}, V_H(N_H(S), S) \geq V_H(0, S_0)$.
- $\exists S^* \in \mathcal{S}$ satisfying $V_H(N_H(S^*), S^*) > V_H(0, S_0)$.

We will say \mathcal{M} is arbitrage-free if \mathcal{H} contains no arbitrage strategies.

The next conditional minmax bounds contemplate the possibility of conditioning on given values of S and trading instance k .

Definition 5 (Conditional Minmax Bounds). *Given a discrete market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$, a node (S, k) and a function Z defined on \mathcal{S} , define*

$$(2.2) \quad \bar{V}_k(S, Z, \mathcal{M}) \equiv \inf_{H \in \mathcal{H}} \sup_{\tilde{S} \in \mathcal{S}_{(S,k)}} [Z(\tilde{S}) - \sum_{i=k}^{N_H(\tilde{S})-1} H_i(\tilde{S}) \Delta_i \tilde{S}].$$

Also define $\underline{V}_k(S, Z, \mathcal{M}) = -\bar{V}_k(S, -Z, \mathcal{M})$.

We use the notation $\bar{V}(S_0, Z, \mathcal{M}) = \bar{V}_0(S, Z, \mathcal{M})$ and $\underline{V}(S_0, Z, \mathcal{M}) = \underline{V}_0(S, Z, \mathcal{M})$. These quantities are called *price bounds*.

Definition 6 (Conditionally 0-Neutral). *We say that a discrete market \mathcal{M} is conditionally 0-neutral at node (S, k) , if*

$$\bar{V}_k(S, Z = 0, \mathcal{M}) = 0.$$

For $k = 0$, we will just say that \mathcal{M} is 0-neutral.

The notion of 0-neutral market, taken from [9], was originally introduced in [7] and considered equivalent to arbitrage-free in their context. In our general setting, it is only a necessary condition for a discrete market to be arbitrage-free [9, Corollary 1] while simultaneously allowing for arbitrage opportunities and a well defined theory of option pricing.

The next definition introduces two basic concepts: a local, and portfolio independent, analogue on \mathcal{S} of the 0-neutral property of \mathcal{M} and a strengthening of this notion representing the local analogue of the arbitrage free property.

Definition 7 (0-Neutral & Arbitrage-Free Nodes). *Given a trajectory space \mathcal{S} and a node (S, j) :*

- (S, j) is called a 0-neutral node if

$$(2.3) \quad \sup_{\tilde{S} \in \mathcal{S}(S, j)} (\tilde{S}_{j+1} - S_j) \geq 0 \text{ and } \inf_{\tilde{S} \in \mathcal{S}(S, j)} (\tilde{S}_{j+1} - S_j) \leq 0.$$

- (S, j) is called an arbitrage-free node if

$$(2.4) \quad \sup_{\tilde{S} \in \mathcal{S}(S, j)} (\tilde{S}_{j+1} - S_j) > 0 \text{ and } \inf_{\tilde{S} \in \mathcal{S}(S, j)} (\tilde{S}_{j+1} - S_j) < 0$$

or

$$(2.5) \quad \sup_{\tilde{S} \in \mathcal{S}(S, j)} (\tilde{S}_{j+1} - S_j) = \inf_{\tilde{S} \in \mathcal{S}(S, j)} (\tilde{S}_{j+1} - S_j) = 0 = (\tilde{S}_{j+1} - S_j).$$

\mathcal{S} is called locally 0-neutral if (2.3) holds at each node (S, j) . \mathcal{S} is said to be locally arbitrage-free if either (2.4) or (2.5) hold at each node (S, j) . If just (2.4) holds at each node, it is said that \mathcal{S} satisfies the up-down property.

A node that satisfies (2.4) will be called an up-down node, and a node satisfying (2.5) will be called a flat node. A node that is 0-neutral but that is not an arbitrage-free node, will be called an arbitrage node.

Observe that the local arbitrage-free property clearly implies the local 0-neutral one. Reference [9, Theorem 2] shows that local 0-neutrality implies 0-neutrality (as per Definition 6) for bounded markets. By [9, Corollary 3] if $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ is a bounded market where \mathcal{S} satisfies the local arbitrage-free property, and N_H are stopping times for any $H \in \mathcal{H}$, then \mathcal{M} is arbitrage free.

2.2. Other Sources of Uncertainty. The formalism just introduced as well as all the developments that follow in the paper can be extended by incorporating another source of uncertainty besides the stock. In order to alleviate the notation in earlier sections, we will only incorporate explicitly this extra variable starting in Section 6. This extra source of uncertainty will be denoted by $W = \{W_i\}$ which, in financial terms, will be considered to be an observable quantity. The sequence elements W_i are assumed to belong to abstract sets Ω_i from which we only require to have defined an equality

relationship. We provide next the simple changes to the previous definitions to accommodate for the new source of uncertainty. The arrow notation \rightarrow indicates how the objects change ((s_0, w_0) are fixed).

$$(2.6) \quad \mathcal{S}_\infty(s_0) \rightarrow \mathcal{S}_\infty^\mathcal{W}(s_0, w_0) \equiv \{\mathbf{S} = \{\mathbf{S}_i \equiv (S_i, W_i)\}_{i \geq 0} : S_i \in \Sigma_i \subset \mathbb{R}, W_i \in \Omega_i, S_0 = s_0, W_0 = w_0\}.$$

$$\mathcal{S} = \mathcal{S}(s_0) \rightarrow \mathcal{S}^\mathcal{W} \equiv \mathcal{S}^\mathcal{W}(s_0, w_0) \subseteq \mathcal{S}_\infty^\mathcal{W}(s_0, w_0).$$

$$H_i(S) \rightarrow H_i(\mathbf{S}).$$

$$\mathcal{S}_{(S,k)} \rightarrow \mathcal{S}_{(\mathbf{S},k)}^\mathcal{W}(s_0, w_0) \equiv \{\tilde{\mathbf{S}} \in \mathcal{S}^\mathcal{W}(s_0, w_0), \tilde{\mathbf{S}}_i = \mathbf{S}_i, 0 \leq i \leq k\}.$$

$$V_H(i, S) \rightarrow V_H(k, \mathbf{S}) = V_H(0, (S_0, w_0)) + \sum_{i=0}^{k-1} H_i(\mathbf{S}) \Delta_i S.$$

Besides the above changes, that concern mostly trajectory sets and the functional dependency $H_i()$ in terms of both variables S_k, W_k , all statements and properties appearing in the paper, only involve the first coordinate S_i (in the tuples (S_i, W_i)) in all algebraic manipulations and are valid in the extended setting. Clearly, H is required to be non-anticipative with respect two both variables S_k and W_k . The new formalism allows W to be used for modeling the unfolding of stock trajectories and so, quantities involved in hedging and intermediate computations will have a dependency on W . We refer to [9] for more precision on this extended/augmented formalism. We will make explicit use of other sources of uncertainty in a good portion of the paper.

3. PRICE INTERVAL IN 0-NEUTRAL MARKETS

This section resumes results from [9] providing conditions on the market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ that guarantee the existence of a price interval for a given function Z defined on \mathcal{S} . The notion of integrability of such functions is replaced by the, so called, minmax functions. Also conditions for the boundedness of $\underline{V}(S_0, Z, \mathcal{M})$ and $\bar{V}(S_0, Z, \mathcal{M})$ are given.

Theorem 1 (Price Interval). *Consider a bounded market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ and a function Z defined on \mathcal{S} ; fix $S \in \mathcal{S}$ and $k \geq 0$. If any node (\tilde{S}, j) , with $\tilde{S} \in \mathcal{S}_{(S,k)}$ and $j \geq k$, is 0-neutral, then*

$$(3.1) \quad \underline{V}_k(S, Z, \mathcal{M}) \leq \bar{V}_k(S, Z, \mathcal{M}).$$

Proof. The result is a special case of Corollary 2 in [9] by noticing that a bounded N_H is also initially bounded (a further condition needed to apply the quoted corollary). Notice that we are assuming portfolios are liquidated at N_H (i.e. $H_k(S) = 0$ for $k \geq N_H(S)$). \square

Under the assumption that $\underline{V}(S, Z, \mathcal{M}) \leq \bar{V}(S, Z, \mathcal{M})$, we will call $[\underline{V}(S_0, Z, \mathcal{M}), \bar{V}(S_0, Z, \mathcal{M})]$ the price interval of Z relative to \mathcal{M} .

The next Theorem gives local conditions ensuring that a discrete market is conditionally 0-neutral. The result is used in the next section to show boundedness of the price bounds as well as in Section 4.

Theorem 2. [9, Theorem 2] *Given a bounded market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$, $k \geq 0$, $S^k \in \mathcal{S}$. If any node (S, j) , $j \geq k$ and $S \in \mathcal{S}_{(S^k, k)}$ is 0-neutral then, \mathcal{M} is conditionally 0-neutral at (S^k, k) .*

The above Theorem holds in more general settings as the bounded assumption on \mathcal{M} can be relaxed in several ways ([9]).

3.1. Minmax functions. Conditions for Boundedness of $\underline{V}(Z)$ and $\bar{V}(Z)$. The integrability conditions, required for payoffs in a probabilistic setting, are replaced in the proposed framework by the, so called, minmax functions (introduced in [9, Definition 14]). In what follows consider a discrete market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$, and a function Z defined on \mathcal{S} .

Definition 8. (*Upper and Lower Minmax Functions*) Given a finite sequence of stopping times $(v_i)_{i=1}^n$ with $v_i < v_{i+1}$ for $1 \leq i < n$, a real sequence $(a_i)_{i=1}^n$, and $b \in \mathbb{R}$, we call Z an upper minmax function if

$$Z(S) \leq \sum_{i=1}^n a_i S_{v_i(S)} + b, \forall S \in \mathcal{S}.$$

Similarly, Z is called a lower minmax function if

$$Z(S) \geq \sum_{i=1}^n a_i S_{v_i(S)} + b, \forall S \in \mathcal{S}.$$

[9, Section 5] provides several examples of options belonging to the class of minmax functions. Proposition 3, in Appendix A, provides conditions for the boundedness of $\bar{V}(Z)$ and $\underline{V}(Z)$.

4. DYNAMIC MINMAX BOUNDS

Arguably, attempting a direct evaluation of the minmax optimization required in (2.2) and in related expressions, is a daunting task. Moreover, the minmax formulation of the problem gives no clues on how to construct the hedging values $H_i(S)$, for a given payoff Z , by means of the unfolding path values S_0, S_1, S_2, \dots

Consider next another pair of numbers, first proposed in [7], namely $\underline{U}_0(S_0, Z, \mathcal{M})$ and $\bar{U}_0(S_0, Z, \mathcal{M})$. These numbers are obtained through a dynamic, or iterative, definition each instance involving a local minmax optimization. Using these definitions we provide conditions under which the global and the iterated definitions coincide.

A special case of the iterative construction was introduced informally in [7] (see also [10] and [8]) and for specific discrete market models. Here we formalize the validity of the approach in such a way that becomes available in a more general class of models and at the same time indicating the differences with the global minmax approach. The references [5] and [6] provide a dynamic programming version of a global minmax optimization. Our approach differs as we make use of specific hypothesis present in our setting.

For simplicity, we introduce the dynamic programming bounds by restricting further the notion of bounded market. We will deal with n -bounded discrete markets:

Definition 9 (*n-Bounded Market*). A market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ is called n -bounded if there exists a constant n so that:

$$\sup_{S \in \mathcal{S}} M(S) \leq n, \text{ where } M(S) \equiv \sup_{H \in \mathcal{H}} N_H(S).$$

We will later introduce further restrictions on the set of portfolios \mathcal{H} . If N_H are stopping times for any $H \in \mathcal{H}$, then so is also M . To see this, let $M \equiv M(S)$, and $S' \in \mathcal{S}_{(S,M)}$; since $N_H(S) \leq M$, for any $H \in \mathcal{H}$, and $S'_i = S_i$ for $0 \leq i \leq M$ then $N_H(S') = N_H(S)$ so $M(S') = M(S)$.

The following inductive definition gives the basic dynamic programming formulation to compute $\bar{V}(S_0, Z, \mathcal{M})$.

Definition 10 (Dynamic Bounds). *Consider an n -bounded, discrete market \mathcal{M} ; for a given function Z defined on \mathcal{S} , any $S \in \mathcal{S}$, and $0 \leq i \leq n$ set*

$$\bar{U}_i(S, Z, \mathcal{M}) = 0 \quad \text{if } i > M(S),$$

$$\bar{U}_{M(S)}(S, Z, \mathcal{M}) = Z(S),$$

and for $0 \leq i < M(S)$,

$$(4.1) \quad \bar{U}_i(S, Z, \mathcal{M}) = \inf_{H \in \mathcal{H}} \left\{ \sup_{S^{i+1} \in \mathcal{S}_{(S,i)}} [\bar{U}_{i+1}(S^{i+1}, Z, \mathcal{M}) - H_i(S) \Delta_i S^{i+1}] \right\}.$$

Also define $\underline{U}_i(S, Z, \mathcal{M}) = -\bar{U}_i(S, -Z, \mathcal{M})$.

Remark 1.

- (1) Since $\underline{U}_0(S, Z, \mathcal{M})$ and $\bar{U}_0(S, Z, \mathcal{M})$ just depend on S_0 , we adopt the notation $\underline{U}_0(S_0, Z, \mathcal{M})$ and $\bar{U}_0(S_0, Z, \mathcal{M})$, respectively.
- (2) Note that in the previous definition

$$H_i(S) = H_i(S^{i+1}), \quad \text{and} \quad \Delta_i S^{i+1} = S_{i+1}^{i+1} - S_i, \quad 0 \leq i < M(S).$$

- (3) From Lemma 4 in Appendix B,

$$(4.2) \quad \begin{aligned} \bar{U}_0(S_0, Z, \mathcal{M}) = & \inf_{H^0 \in \mathcal{H}} \left\{ \sup_{S^1 \in \mathcal{S}} [-H_0^0(S^1) \Delta_0 S^1 + \inf_{H^1 \in \mathcal{H}} \left\{ \sup_{S^2 \in \mathcal{S}_{(S^1,1)}} [-H_1^1(S^2) \Delta_1 S^2 + \cdots + \right. \right. \\ & \left. \left. + \inf_{H^{n-1} \in \mathcal{H}} \left\{ \sup_{S^n \in \mathcal{S}_{(S^{n-1},n-1)}} [Z(S^n) - H_{n-1}^{n-1}(S^n) \Delta_{n-1} S^n] \right\} \cdots \right] \right\} \right\}. \end{aligned}$$

Remark 2. Assume N_H is stopping time for any $H \in \mathcal{H}$ and fix $S \in \mathcal{S}$ and $0 \leq i \leq n$:

$$\text{if } S' \in \mathcal{S}_{(S,i)} \text{ then } \bar{U}_i(S', Z, \mathcal{M}) = \bar{U}_i(S, Z, \mathcal{M}).$$

Since we already have noted that M is a stopping time then $\bar{U}_M(S, Z, \mathcal{M}) = \bar{U}_M(S', Z, \mathcal{M})$ if $S' \in \mathcal{S}_{(S,M)}$ and where $M = M(S)$. Moreover, for $S' \in \mathcal{S}_{(S,i)}$, $M(S) < i$ if and only if $M(S') = M(S) < i$. Thus $\bar{U}_i(S, Z, \mathcal{M}) = \bar{U}_i(S', Z, \mathcal{M})$ if $i \geq M(S)$. Reasoning inductively, assume $i < M(S)$ and that the statement to be proven is valid for $j > i$, consequently $i < M(S')$. Then

$$\begin{aligned} \bar{U}_i(S', Z, \mathcal{M}) &= \inf_{H \in \mathcal{H}} \left\{ \sup_{\tilde{S} \in \mathcal{S}_{(S',i)}} [\bar{U}_{i+1}(\tilde{S}, Z, \mathcal{M}) - H_i(S') \Delta_i \tilde{S}] \right\} = \\ &= \inf_{H \in \mathcal{H}} \left\{ \sup_{\tilde{S} \in \mathcal{S}_{(S,i)}} [\bar{U}_{i+1}(\tilde{S}, Z, \mathcal{M}) - H_i(S) \Delta_i \tilde{S}] \right\} = \bar{U}_i(S, Z, \mathcal{M}). \end{aligned}$$

Remark 3. For any $S \in \mathcal{S}$ and $0 \leq k < M(S)$, define

$$(4.3) \quad I_S^k \equiv \{H_k(S) : H \in \mathcal{H}\}.$$

In the sequel, we will make use of the following rewrite of the expression in (4.1).

For $0 \leq k < M(S)$,

$$(4.4) \quad \bar{U}_k(S, Z, \mathcal{M}) = \inf_{u \in I_S^k} \left\{ \sup_{S' \in \mathcal{S}_{(S,k)}} [\bar{U}_{k+1}(S', Z, \mathcal{M}) - u \Delta_k S'] \right\}.$$

Observe that if $S' \in \mathcal{S}_{(S,k)}$, then $\mathcal{S}_{(S',k)} = \mathcal{S}_{(S,k)}$ and $I_{S'}^k = I_S^k$.

We have the following general relationship between the global and dynamic bounds.

Proposition 1. *For any function Z defined on a discrete n -bounded market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$, the following inequality holds:*

$$(4.5) \quad \bar{U}_0(S_0, Z, \mathcal{M}) \leq \bar{V}(S_0, Z, \mathcal{M}),$$

and hence $\underline{U}_0(S_0, Z, \mathcal{M}) \geq \underline{V}(S_0, Z, \mathcal{M})$ is also valid.

Proof. Fix $H \in \mathcal{H}$, using $H^0 \equiv H$ in (4.2) it follows that

$$\begin{aligned} \bar{U}_0(S_0, Z, \mathcal{M}) &\leq \sup_{S^1 \in \mathcal{S}} [-H_0(S^1) \Delta_0 S^1 + \inf_{H^1 \in \mathcal{H}} \left\{ \sup_{S^2 \in \mathcal{S}_{(S^1,1)}} [-H_1^1(S^1) \Delta_1 S^2 + \dots + \right. \\ &\quad \left. + \inf_{H^{n-1} \in \mathcal{H}} \left\{ \sup_{S^n \in \mathcal{S}_{(S^{n-1},n-1)}} [Z(S^n) - H_{n-1}^{n-1}(S^{n-1}) \Delta_{n-1} S^n] \dots \right\} \right\}]. \end{aligned}$$

Recursively, setting $H^i \equiv H$, $2 \leq i \leq n-1$, we obtain the following inequality:

$$(4.6) \quad \begin{aligned} \bar{U}_0(S, Z, \mathcal{M}^n) &\leq \sup_{S^1 \in \mathcal{S}} [-H_0(S^1) \Delta_0 S^1 + \sup_{S^2 \in \mathcal{S}_{(S^1,1)}} [-H_1(S^1) \Delta_1 S^2 + \dots + \\ &\quad + \sup_{S^n \in \mathcal{S}_{(S^{n-1},n-1)}} [Z(S^n) - H_{n-1}(S^{n-1}) \Delta_{n-1} S^n] \dots]. \end{aligned}$$

The result then follows by using (4.6) in combination with (B.1) from Appendix B. \square

The next corollary, being a consequence of Theorem 2 and Proposition 1, represents the dynamic analogous of the 0-neutral condition,

Corollary 1. *Let $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ a discrete n -bounded market model and $Z \geq 0$ a function defined on \mathcal{S} . If \mathcal{S} satisfies the local 0-neutral property, then:*

- (1) *For any $S \in \mathcal{S}$, and $0 \leq i \leq n$, $\bar{U}_i(S, Z, \mathcal{M}) \geq 0$.*
- (2) *$\underline{U}_0(S_0, Z = 0, \mathcal{M}) = \bar{U}_0(S_0, Z = 0, \mathcal{M}) = 0$.*

Proof. For (1) we proceed by induction backwards, since

$$\bar{U}_n(S, Z, \mathcal{M}) = Z(S) \geq 0 \text{ or } \bar{U}_n(S, Z, \mathcal{M}) = 0$$

this is so by definition as for any $S \in \mathcal{S}$, $M(S) \equiv \sup_{H \in \mathcal{H}} N_H(S) \leq n$.

Assume $\bar{U}_{i+1}(S, Z, \mathcal{M}) \geq 0$, for some $0 \leq i \leq n-1$ and any $S \in \mathcal{S}$. For fixed S , if $i \geq M(S)$ then $\bar{U}_i(S, Z, \mathcal{M}) = 0$ or $\bar{U}_i(S, Z, \mathcal{M}) = Z(S) \geq 0$. On the other hand, if $i < M(S)$, since \mathcal{S} satisfies the local 0-neutral property at (S, i) , for any $H \in \mathcal{H}$

$$\sup_{\tilde{S} \in \mathcal{S}_{(S,i)}} [-H_i(S) \Delta_i \tilde{S}] \geq 0.$$

Indeed, if $H_i(S) = 0$ then, $\sup_{\tilde{S} \in \mathcal{S}_{(S,i)}} [-H_i(S) \Delta_i \tilde{S}] = 0$. If $H_i(S) > 0$, by (2.3),

$$\sup_{\tilde{S} \in \mathcal{S}_{(S,i)}} [-H_i(S) \Delta_i \tilde{S}] \geq 0,$$

and this is also true if $-H_i(S) > 0$. Thus $\sup_{\tilde{S} \in \mathcal{S}_{(S,i)}} [\overline{U}_{i+1}(\tilde{S}, Z, \mathcal{M}^n) - H_i(S) \Delta_i \tilde{S}] \geq 0$ and then

$$\overline{U}_i(S, Z, \mathcal{M}^n) \geq 0.$$

Statement (2) follows from Proposition 1 and item (1) above since

$$0 \leq \overline{U}_0(S_0, Z = 0, \mathcal{M}^n) \leq \overline{V}(S_0, Z = 0, \mathcal{M}) = 0,$$

where the last equality follows from Theorem 2. \square

4.1. Full Set of Portfolios. We are interested in obtaining the reverse of inequality (4.5). To achieve that goal, it will be necessary to introduce some conditions on the set of portfolios, as well as other conditions, that imply equality in the inequality (4.5) and also lead to an efficient method to compute the dynamic bounds. Results in [5] suggest that having all possible portfolios may lead to establishing the desired equality; this motivates the following definition.

Definition 11 (FULL Set of Portfolios). *Given a discrete market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$, consider $k \geq 0$, $S^* \in \mathcal{S}$, $j \geq k$ and range set,*

$$I_{\mathcal{S}_{(S^*,k)}}^j \equiv \{\tilde{H}_j(S) : \tilde{H} \in \mathcal{H}, S \in \mathcal{S}_{(S^*,k)}\}.$$

We will say that \mathcal{H} is FULL, if the set of functions with domain $\mathcal{S}_{(S^,k)}$ and range $I_{\mathcal{S}_{(S^*,k)}}^j$, which are non-anticipative with respect to the first j coordinates of its argument, coincides, for any such k, S^* and j , with the set of functions $H_j|_{\mathcal{S}_{(S^*,k)}} : \mathcal{S}_{(S^*,k)} \rightarrow \mathbb{R}$ where $H \in \mathcal{H}$.*

Observe that $\mathcal{S}_{(S,k)} = \mathcal{S}_{(S^*,k)}$ for $S \in \mathcal{S}_{(S^*,k)}$, then the notation $I_{\mathcal{S}_{(S^*,k)}}^j$ is justified. A useful, particular possibility is that $I_{\mathcal{S}_{(S,k)}}^j \equiv \mathbb{R}$, for any $k \geq 0$, and $S \in \mathcal{S}$.

The following theorem shows that the condition *FULL* on the set of portfolios assures the equality in (4.5). Observe that that condition is natural, in the sense that any of the values $H_j(S)$, taken by a portfolio $H \in \mathcal{H}$, at a rebalancing instance j , for some $S \in \mathcal{S}$, should also be taken at any $S^* \in \mathcal{S}_{(S,k)}$ if $j \geq k$. It means that there exists $H^* \in \mathcal{H}$ such that $H_j^*(S^*) = H_j(S)$. So, any set of portfolios \mathcal{H} can be extended to a set $\overline{\mathcal{H}}$ which is *FULL* as we explain next.

For $j \geq k$ and $h : \mathcal{S}_{(S^*,k)} \rightarrow I_{\mathcal{S}_{(S^*,k)}}^j$, non-anticipative with respect to the first j coordinates of its argument, define

$$H_j(S) = h(S) \quad \text{for } S \in \mathcal{S}_{(S^*,k)}.$$

This is a well defined function by non-anticipativity and because $\mathcal{S}_{(S^*,k)}$ is a partition of \mathcal{S} . Defining $H_j(S) = 0$ for $j < k$, the collection of functions $H = \{H_j\}_{j \geq 0}$ are non-anticipative. Moreover for any S and j there exist $H^{S,j}$ (actually the same for all $\mathcal{S}_{(S,j)}$) such that $h(S) = H_j^{S,j}(S)$; we can then set

$$N_H(S) = \max\{N_{H^{S,j}}(S) : S, j\}.$$

Which is well defined if \mathcal{M} is n -bounded.

Theorem 3. For a general n -bounded market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$, where \mathcal{H} is FULL, and for a given function Z defined on \mathcal{S} , we have

$$(4.7) \quad \bar{V}(S_0, Z, \mathcal{M}) = \bar{U}_0(S_0, Z, \mathcal{M}).$$

Proof. We proceed by induction on n . For $n = 1$, we have from (2.2) and Definition 10 that

$$\bar{V}(S_0, Z, \mathcal{M}) = \inf_{H \in \mathcal{H}} \sup_{S \in \mathcal{S}} [Z(S) - H_0(S)(S_1 - S_0)] = \bar{U}_0(S_0, Z, \mathcal{M}).$$

Let us now assume that the theorem holds for every n -bounded discrete market model for which \mathcal{H} is FULL, and consider an $(n+1)$ -bounded one, $\mathcal{M} = \mathcal{S} \times \mathcal{H}$. By Lemma 3, item (2), it follows that

$$\bar{U}_0(S_0, Z, \mathcal{M}) = \inf_{H^0 \in \mathcal{H}} \sup_{S^1 \in \mathcal{S}} \{-H_0^0(S^1)(S_1^1 - S_0) + \bar{U}_0(\hat{S}_0, \hat{Z}, \widehat{\mathcal{M}}_1)\}.$$

Where $\widehat{\mathcal{M}}_1, \hat{Z}, \hat{S}_0$, are introduced by Definition 17 (this definition and quoted lemmas are located in Appendix B). By Lemma 3, item (1), and Lemma 5, $\widehat{\mathcal{M}}_1$ is a n -bounded discrete market model, and \mathcal{H} is FULL, then, by inductive hypothesis we have,

$$(4.8) \quad \bar{U}_0(S_0, Z, \mathcal{M}) = \inf_{H^0 \in \mathcal{H}} \sup_{S^1 \in \mathcal{S}} \{-H_0^0(S^1)(S_1^1 - S_0) + \bar{V}_0(\hat{S}_0, \hat{Z}, \widehat{\mathcal{M}}_1)\}.$$

Recall that $\bar{V}_0(\hat{S}_0, \hat{Z}, \widehat{\mathcal{M}}_1)$, in the last expression, depends on each selection of S^1 ; it can be written as,

$$\begin{aligned} \bar{V}_0(\hat{S}_0, \hat{Z}, \widehat{\mathcal{M}}_1) &= \inf_{\hat{H}^1 \in \widehat{\mathcal{H}}} \sup_{\hat{S}^2 \in \widehat{\mathcal{S}}} [\hat{Z}(\hat{S}^2) - \sum_{i=0}^{n-1} \hat{H}_i^1(\hat{S}^2) \Delta_i \hat{S}^2] = \\ &= \inf_{H^1 \in \mathcal{H}} \sup_{S^2 \in \mathcal{S}_{(S^1, 1)}} [Z(S^2) - \sum_{i=1}^n H_i^1(S^2) \Delta_i S^2]. \end{aligned}$$

Because Proposition 1 we only need to prove the inequality,

$$\bar{V}_0(S_0, Z, \mathcal{M}) \leq \bar{U}_0(S_0, Z, \mathcal{M}).$$

Thus, we can assume that $\bar{U}_0(S_0, Z, \mathcal{M}) > -\infty$, and consequently for $S^1 \in \mathcal{S}$, $\bar{V}_0(\hat{S}_0, \hat{Z}, \widehat{\mathcal{M}}_1) > -\infty$. Fix $\varepsilon > 0$, then for each conditional space of trajectories: $\mathcal{S}_{(S^1, 1)}$ there exists $H^{S^1} \in \mathcal{H}$, such that

$$\sup_{S^2 \in \mathcal{S}_{(S^1, 1)}} [Z(S^2) - \sum_{i=1}^n H_i^{S^1}(S^2) \Delta_i S^2] < \varepsilon + \bar{V}_0(\hat{S}_0, \hat{Z}, \widehat{\mathcal{M}}_1).$$

Then for $H^0 \in \mathcal{H}$

$$(4.9) \quad -H_0^0(S^1) \Delta_0 S^1 + \sup_{S^2 \in \mathcal{S}_{(S^1, 1)}} [Z(S^2) - \sum_{i=1}^n H_i^{S^1}(S^2) \Delta_i S^2] < \varepsilon - H_0^0(S^1) \Delta_0 S^1 + \bar{V}_0(\hat{S}_0, \hat{Z}, \widehat{\mathcal{M}}_1).$$

Since \mathcal{H} is FULL, there exist $H^\varepsilon \in \mathcal{H}$ such that, $H_0^\varepsilon = H_0^0$ and for any $S^1 \in \mathcal{S}$

$$H_i^\varepsilon(S) = H_i^{S^1}(S), \text{ if } S \in \mathcal{S}_{(S^1, 1)} \quad \text{and} \quad i \geq 1.$$

The functions H_i^ε are well defined, since the family $\{\mathcal{S}_{(S^1, 1)}\}_{S^1 \in \mathcal{S}}$ is a partition of \mathcal{S} . Now, from (4.9), we have

$$\sup_{S^1 \in \mathcal{S}} \sup_{S^2 \in \mathcal{S}_{(S^1, 1)}} [Z(S^2) - \sum_{i=0}^n H_i^\varepsilon(S^2) \Delta_i S^2] < \varepsilon + \sup_{S^1 \in \mathcal{S}} \{-H_0^0(S^1) \Delta_0 S^1 + \bar{V}_0(\hat{S}_0, \hat{Z}, \widehat{\mathcal{M}}_1)\},$$

and from (4.8),

$$\begin{aligned}
\bar{V}_0(S_0, Z, \mathcal{M}) &= \inf_{H^0 \in \mathcal{H}} \sup_{S^1 \in \mathcal{S}} [Z(S^1) - \sum_{i=0}^n H_i^0(S) \Delta_i S^1] \\
&\leq \varepsilon + \inf_{H^0 \in \mathcal{H}} \sup_{S^1 \in \mathcal{S}} \{-H_0^0(S^1) \Delta_0 S^1 + \bar{V}_0(\hat{S}_0, \hat{Z}, \widehat{\mathcal{M}}_1)\} \\
&= \varepsilon + \bar{U}_0(S_0, Z, \mathcal{M}).
\end{aligned}$$

□

There are other conditions on the set of portfolios which lead to the equality (4.7), we present an example next.

4.2. u-Complete Set of Portfolios. We introduce another condition that allows to have the equality $\bar{U}_0(S_0, Z, \mathcal{M}) = \bar{V}_0(S_0, Z, \mathcal{M})$. Most of the proofs and some new notation that is required for this subsection, is included in Appendix B.2.

Definition 12 (*u-Complete Market*). *We will say that an n -bounded discrete market \mathcal{M} is u-complete with respect to a real function Z defined on \mathcal{S} , if for any $S \in \mathcal{S}$, and $1 \leq k < M(S)$, there exists $H^* \in \mathcal{H}$, verifying*

$$(4.10) \quad \bar{U}_k(S, Z, \mathcal{M}) = \sup_{S' \in \mathcal{S}_{(S,k)}} [U_{k+1}(S', Z, \mathcal{M}) - H_k^*(S) \Delta_k S'].$$

Remark 4. *Recalling Remark 2, if N_H is a stopping time, equation (4.10) is valid for any $S^* \in \mathcal{S}_{(S,k)}$, in the sense that $\bar{U}_k(S^*, Z, \mathcal{M})$ can be the left member of the equality.*

Theorem 4 (Proof in Appendix B.2). *If $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ is an n -bounded discrete market u-complete with respect to a given function Z defined on \mathcal{S} , then*

$$\bar{V}(S_0, Z, \mathcal{M}) = \bar{U}_0(S_0, Z, \mathcal{M}).$$

Taking together, Lemma 1 and Theorem 5 below provide practical and useful hypothesis for an application of Theorem 4 above.

Lemma 1 (Proof in Appendix B.2). *Consider an n -bounded discrete market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ and node (S^*, k) with $0 \leq k < M(S^*)$. If $I_{S^*}^k \equiv \{H_k(S^*) : H \in \mathcal{H}\}$ given by (4.3) is a compact subset of \mathbb{R} or \mathcal{S} satisfies the up-down property (as per Definition 7) at (S^*, k) and $I_{S^*}^k = \mathbb{R}$, then there exists $u^* \in I_{S^*}^k$, verifying for any $S \in \mathcal{S}_{(S^*, k)}$ that*

$$(4.11) \quad \inf_{u \in I_{S^*}^k} \sup_{S' \in \mathcal{S}_{(S,k)}} [\bar{U}_{k+1}(S', Z, \mathcal{M}) - u \Delta_k S'] = \sup_{S' \in \mathcal{S}_{(S,k)}} [\bar{U}_{k+1}(S', Z, \mathcal{M}) - u^* \Delta_k S'].$$

Moreover, in the case $I_{S^*}^k = \mathbb{R}$, there exists $R > 0$ such that $|u^*| \leq R$.

If all N_H are stopping times (and so is M as well), $S \in \mathcal{S}_{(S^*, k)}$ and $k < M(S^*)$ then, the left side of (4.11) is $\bar{U}_k(S, Z, \mathcal{M})$.

Theorem 5. *Assume $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ is an n -bounded discrete market, such that N_H is a stopping time for any $H \in \mathcal{H}$. Also assume that for any $S \in \mathcal{S}$ and $0 \leq k < M(S)$, the sets I_S^k introduced by (4.3) and \mathcal{S} verify the hypothesis of Lemma 1. Define*

$$H_k^* : \mathcal{S} \rightarrow \cup_{S \in \mathcal{S}} I_S^k \quad \text{by} \quad H_k^*(S') \equiv u^* \quad \text{for any } S' \in \mathcal{S}_{(S,k)},$$

where u^* is given by Lemma 1 with $S^* = S$ and k and verifies (4.11). Also define

$$(4.12) \quad H^* = (H_i^*)_{i \geq 0} \text{ where } H_i^* = 0 \text{ for } i \geq M(S), \quad N_{H^*}(S) = M(S), \text{ and } V_{H^*}(0, s_0) = H_0^*(S)s_0.$$

Then, with $\mathcal{H}^* = \mathcal{H} \cup \{H^*\}$, $\mathcal{M}^* = \mathcal{S} \times \mathcal{H}^*$ is an u -complete discrete market.

Proof. First it is necessary to show that H^* defined by (4.12) is non-anticipative. Let $S, S' \in \mathcal{S}$ with $S_i = S'_i$ for $0 \leq i \leq k$, then $\mathcal{S}_{(S,k)} = \mathcal{S}_{(S',k)}$ and $I_S^k = I_{S'}^k$, so $H_k^*(S') = u^* = H_k^*(S)$.

For the u -completion of \mathcal{M}^* , we first prove by backward induction that $\bar{U}_i(S, Z, \mathcal{M}) = \bar{U}_i(S, Z, \mathcal{M}^*)$ for any $0 \leq i \leq n$.

Since for any $S \in \mathcal{S}$, $M^*(S) = \max(\{N_H(S) : H \in \mathcal{H}\} \cup \{N_{H^*}(S)\}) = M(S)$, it is clear that $\bar{U}_n(S, Z, \mathcal{M}) = \bar{U}_n(S, Z, \mathcal{M}^*)$. Assume by inductive hypothesis that $\bar{U}_i(S, Z, \mathcal{M}) = \bar{U}_i(S, Z, \mathcal{M}^*)$ for any $j+1 \leq i \leq n$ and observe that $I_S^{*j} = \{H_j(S) : H \in \mathcal{H}\} \cup \{H_j^*(S)\} = I_S^j$. Notice that the fact that N_H is a stopping time for any $H \in \mathcal{H}$ implies N_{H^*} is also a stopping time. Therefore, using (4.4) we obtain,

$$\begin{aligned} \bar{U}_j(S, Z, \mathcal{M}^*) &= \inf_{u \in I_S^{*j}} \left\{ \sup_{S' \in \mathcal{S}_{(S,j)}} [\bar{U}_{j+1}(S', Z, \mathcal{M}^*) - u \Delta_j S] \right\} = \\ &= \inf_{u \in I_S^j} \left\{ \sup_{S' \in \mathcal{S}_{(S,j)}} [\bar{U}_{j+1}(S', Z, \mathcal{M}) - u \Delta_j S] \right\} = \bar{U}_j(S, Z, \mathcal{M}). \end{aligned}$$

Finally for (4.11), for any $k \geq 0$,

$$\bar{U}_k(S, Z, \mathcal{M}^*) = \bar{U}_k(S, Z, \mathcal{M}) = \sup_{S' \in \mathcal{S}_{(S,k)}} [\bar{U}_{k+1}(S', Z, \mathcal{M}) - H_k^*(S) \Delta_k \tilde{S}],$$

with $H^* \in \mathcal{H}^*$. □

5. CONVEX ENVELOPE FOR DYNAMIC MINIMAX BOUNDS

This section describes how to calculate the dynamic bounds $\bar{U}_i(S, Z, \mathcal{M})$ introduced in the previous section. In what follows, we will assume that the dynamic bounds are finite, this, for example, follows by an application of Proposition 1 or under the assumptions of Proposition 4.

We will consider an n -bounded discrete market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ (as per Definition 9), with \mathcal{H} FULL. Then, under these conditions, $\bar{U}_0(S_0, Z, \mathcal{M}) = \bar{V}(S_0, Z, \mathcal{M})$.

For $S \in \mathcal{S}$, and $0 < i < M(S)$ we are going to indicate a geometric procedure, originally introduced in [7] for a specific example, in order to compute (4.4):

$$\bar{U}_i(S, Z, \mathcal{M}) = \inf_{u_i \in I_S^i} \sup_{S' \in \mathcal{S}_{(S,i)}} \{ \bar{U}_{i+1}(S', Z, \mathcal{M}) - u_i(S'_{i+1} - S_i) \}.$$

For an arbitrary, but momentarily fixed, $S' \in \mathcal{S}_{(S,i)}$, set

$$\ell(x) = \bar{U}_{i+1}(S', Z, \mathcal{M}) - u_i(S'_{i+1} - x),$$

i.e. the line in the plane, through the point $(S'_{i+1}, \bar{U}_{i+1}(S', Z, \mathcal{M}))$ with slope u_i . Thus,

$$\bar{U}_{i+1}(S', Z, \mathcal{M}) - u_i(S'_{i+1} - S_i)$$

is the intersection of ℓ with the vertical straight line $x = S_i$. Therefore, for each fixed $u_i \in I_S^i$, with some abuse of language

$$\sup_{S' \in \mathcal{S}_{(S,i)}} \{ \bar{U}_{i+1}(S', Z, \mathcal{M}) - u_i(S'_{i+1} - S_i) \}$$

is the largest of the ordinates of the points of intersection between the straight lines ℓ and $x = S_i$. Then $\overline{U}_i(S, Z, \mathcal{M})$ becomes the lowest value of these largest intersections.

To complete the geometric procedure to compute $\overline{U}_i(S, Z, \mathcal{M})$, assume that both of the following sets are nonempty,

$$\mathcal{S}_{(S,i)}^- = \{S' \in \mathcal{S}_{(S,i)} : S'_{i+1} \leq S_i\}, \text{ and } \mathcal{S}_{(S,i)}^+ = \{S' \in \mathcal{S}_{(S,i)} : S'_{i+1} > S_i\}.$$

The sets can also be defined interchanging the strict inequality. For $S^+ \in \mathcal{S}_{(S,i)}^+$ and $S^- \in \mathcal{S}_{(S,i)}^-$ lets call $u_{(S^+, S^-)}$ the slope of the straight line in the plane through the points $(S_{i+1}^+, \overline{U}_{i+1}(S^+, Z, \mathcal{M}))$ and $(S_{i+1}^-, \overline{U}_{i+1}(S^-, Z, \mathcal{M}))$. Next we are going to show that

$$(5.1) \quad \overline{U}_i(S, Z, \mathcal{M}) = \sup_{\substack{S^+ \in \mathcal{S}_{(S,i)}^+ \\ S^- \in \mathcal{S}_{(S,i)}^-}} \{\overline{U}_{i+1}(S^+, Z, \mathcal{M}) - u_{(S^+, S^-)}(S_{i+1}^+ - S_i)\} \equiv L_i(S, Z, \mathcal{M}).$$

That is, $\overline{U}_i(S, Z, \mathcal{M})$, is the largest intersection of the referred lines with the line $x = S_i$.

Remark 5. For any $S^+ \in \mathcal{S}_{(S,i)}^+$ and $S^- \in \mathcal{S}_{(S,i)}^-$

$$\overline{U}_{i+1}(S^+, Z, \mathcal{M}) - u_{(S^+, S^-)}(S_{i+1}^+ - S_i) = \overline{U}_{i+1}(S^-, Z, \mathcal{M}) - u_{(S^+, S^-)}(S_{i+1}^- - S_i).$$

Proposition 2. Let $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ an n -bounded discrete market. Then, for all $S \in \mathcal{S}$,

$$L_i(S, Z, \mathcal{M}) \leq \overline{U}_i(S, Z, \mathcal{M}).$$

Proof. We consider first the case $L_i(S, Z, \mathcal{M}) < \infty$. Let $\delta > 0$, then there is $S^{+, \delta} \in \mathcal{S}_{(S,i)}^+$ and $S^{-, \delta} \in \mathcal{S}_{(S,i)}^-$ such that

$$L_i(S, Z, \mathcal{M}) \leq \overline{U}_{i+1}(S^{+, \delta}, Z, \mathcal{M}) - u_{(S^{+, \delta}, S^{-, \delta})}(S_{i+1}^{+, \delta} - S_i) + \delta.$$

For $u_i \in I_S^i$ such that $u_i \leq u_{(S^{+, \delta}, S^{-, \delta})}$,

$$L_i(S, Z, \mathcal{M}) \leq \overline{U}_{i+1}(S^{+, \delta}, Z, \mathcal{M}) - u_i(S_{i+1}^{+, \delta} - S_i) + \delta \leq \sup_{S' \in \mathcal{S}_{(S,i)}} \{\overline{U}_{i+1}(S', Z, \mathcal{M}) - u_i(S'_{i+1} - S_i)\} + \delta.$$

On the other hand, if $u_i > u_{(S^{+, \delta}, S^{-, \delta})}$, observing that by Remark 5,

$$\begin{aligned} L_i(S, Z, \mathcal{M}) &\leq \overline{U}_{i+1}(S^{-, \delta}, Z, \mathcal{M}) - u_{(S^{+, \delta}, S^{-, \delta})}(S_{i+1}^{-, \delta} - S_i) + \delta \leq \\ &\overline{U}_{i+1}(S^{-, \delta}, Z, \mathcal{M}) - u_i(S_{i+1}^{-, \delta} - S_i) + \delta \leq \sup_{S' \in \mathcal{S}_{(S,i)}} \{\overline{U}_{i+1}(S', Z, \mathcal{M}) - u_i(S'_{i+1} - S_i)\} + \delta. \end{aligned}$$

Then

$$L_i(S, Z, \mathcal{M}) \leq \sup_{S' \in \mathcal{S}_{(S,i)}} \{\overline{U}_{i+1}(S', Z, \mathcal{M}) - u_i(S'_{i+1} - S_i)\} + \delta,$$

for all $u_i \in I_S^i$ and for all $\delta > 0$. Therefore

$$L_i(S, Z, \mathcal{M}) \leq \inf_{u_i \in I_S^i} \sup_{S' \in \mathcal{S}_{(S,i)}} [\overline{U}_{i+1}(S', Z, \mathcal{M}) - u_i(S'_{i+1} - S_i)] = \overline{U}_i(S, Z, \mathcal{M}).$$

Assume now $L_i(S, Z, \mathcal{M}) = \infty$. Then, for all $B \in \mathbb{R}$ there exist $S^+ \in \mathcal{S}_{(S,i)}^+$, $S^- \in \mathcal{S}_{(S,i)}^-$ such that

$$B \leq \overline{U}_{i+1}(S^+, Z, \mathcal{M}) - u_{(S^+, S^-)}(S_{i+1}^+ - S_i).$$

With a similar reasoning than before it follows that $B \leq \sup_{S' \in \mathcal{S}_{(S,i)}} \{\overline{U}_{i+1}(S', Z, \mathcal{M}) - u_i(S'_{i+1} - S_i)\}$, for all $B \in \mathbb{R}$ and $u_i \in I_S^i$, therefore $L_i(S, Z, \mathcal{M}) = \overline{U}_i(S, Z, \mathcal{M}) = \infty$.

□

The next Theorem, which requires extra assumptions, will give us an easier way to solve the optimization problem for $I_S^i = \mathbb{R}$. This will allow us to present more efficient algorithms. The assumption $I_S^i = \mathbb{R}$ is a convenient way of guaranteeing $u_{(S^\bullet, S^\circ)} \in I_S^i$.

Theorem 6 (Proof in Appendix C). *Let $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ a n -bounded discrete market. If for any $S \in \mathcal{S}$, $I_S^i = \mathbb{R}$, and either one the two following conditions for $S \in \mathcal{S}$ below hold,*

- (1) $L_i(S, Z, \mathcal{M}) = \bar{U}_{i+1}(S^\bullet, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^\bullet - S_i)$ for some $S^\bullet \in \mathcal{S}_{(S,i)}^+$, $S^\circ \in \mathcal{S}_{(S,i)}^-$.
- (2) For any $S' \in \mathcal{S}_{(S,i)}$, $0 < a \leq |S'_{i+1} - S_i| \leq b$ (a and b may depend on S).

Then,

$$\bar{U}_i(S, Z, \mathcal{M}) = L_i(S, Z, \mathcal{M}).$$

As a special case of Theorem 6 we can obtain a result from [8] for European convex payoffs. An European option defined in a market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ will be called convex, if its payoff function Z is given by a convex real variable function Z^f , as $Z(S) = Z^f(S_{M(S)})$ for any $S \in \mathcal{S}$.

Notice that the parameters u and d appearing in the next corollary could depend on S_0, \dots, S_i .

Corollary 2. *Let $0 < d < 1 < u$. Consider a n -bounded market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ in which the stock trajectories satisfy the constraints:*

$$\forall S \in \mathcal{S} \text{ and } 0 \leq i \leq M(S) - 1, S_{i+1} \in [dS_i, uS_i].$$

Let $Z(S) = Z^f(S_{M(S)})$ be the payoff function of an European derivative, and assume that Z^f is convex and $M(\cdot)$ a stopping time. Then, for all $S \in \mathcal{S}$ and $0 \leq i < M(S)$ the dynamic bounds are convex and generated by

$$\bar{U}_i(S, Z, \mathcal{M}) = \frac{1-d}{u-d} \bar{U}_{i+1}(S^u, Z, \mathcal{M}) + \frac{u-1}{u-d} \bar{U}_{i+1}(S^d, Z, \mathcal{M})$$

where $S^u, S^d \in \mathcal{S}_{(S,i)}$ such that $S_{i+1}^u = u S_i$ and $S_{i+1}^d = d S_i$.

Proof. Let $S \in \mathcal{S}$ such that $M(S) = n$. Then, by Theorem 6,

$$\bar{U}_{n-1}(S, Z, \mathcal{M}) = \sup_{\substack{S^+ \in \mathcal{S}_{(S,n-1)}^+ \\ S^- \in \mathcal{S}_{(S,n-1)}^-}} \{ \bar{U}_n(S^+, Z, \mathcal{M}) - u_{(S^+, S^-)}(S_n^+ - S_{n-1}) \}.$$

Since \mathcal{M} is an n -bounded market and M is a stopping time, $M(\tilde{S}) = n$ for all $\tilde{S} \in \mathcal{S}_{(S,n-1)}$. Then $\bar{U}_n(\tilde{S}, Z, \mathcal{M}) = Z(\tilde{S}) = Z^f(\tilde{S}_n)$. On the other hand, since $S_n^u \geq S_n^+$ for any $S^+ \in \mathcal{S}_{(S,n-1)}^+$ and Z is convex,

$$Z^f(S_n^u) - u_{(S^u, S^d)}(S_n^u - S_n^+) \geq Z^f \left(S_n^u \left(1 - \frac{S_n^u - S_n^+}{S_n^u - S_n^d} \right) + S_n^d \left(\frac{S_n^u - S_n^+}{S_n^u - S_n^d} \right) \right) = Z^f(S_n^+)$$

Similarly, since $S_n^u \geq S_n^-$ for any $S^- \in \mathcal{S}_{(S,n-1)}^-$ and Z is convex, it follows

$$Z^f(S_n^u) - u_{(S^u, S^d)}(S_n^u - S_n^-) \geq Z^f \left(S_n^u \left(1 - \frac{S_n^u - S_n^-}{S_n^u - S_n^d} \right) + S_n^d \left(\frac{S_n^u - S_n^-}{S_n^u - S_n^d} \right) \right) = Z^f(S_n^-)$$

Then by Lemma 7 in Appendix D,

$$Z^f(S_n^+) - u_{(S^+, S^-)}(S_n^+ - S_{n-1}) \leq Z^f(S_n^u) - u_{(S^u, S^d)}(S_n^u - S_{n-1}),$$

for all $S^+ \in \mathcal{S}_{(S,n-1)}^+$ and $S^- \in \mathcal{S}_{(S,n-1)}^-$. Therefore

$$\begin{aligned} \overline{U}_{n-1}(S, Z, \mathcal{M}) &= Z^f(S_n^u) - u_{(S^u, S^d)}(S_n^u - S_{n-1}) = \\ &= Z^f(uS_{n-1}) - \left(\frac{Z^f(uS_{n-1}) - Z^f(dS_{n-1})}{uS_{n-1} - dS_{n-1}} \right) (uS_{n-1} - S_{n-1}) = \frac{1-d}{u-d} Z^f(uS_{n-1}) + \frac{u-1}{u-d} Z^f(dS_{n-1}). \end{aligned}$$

Since the property of convexity is preserved under scaling and under taking positive linear combinations, it is seen from the above that $\overline{U}_{n-1}(S, Z, \mathcal{M})$ is convex and only depends on the value of S_{n-1} .

We proceed now by backward induction; let $0 \leq i < n$ and suppose that $\overline{U}_{i'}(S, Z, \mathcal{M})$ is convex and generated by (5.2) for all $i < i' < n$. So $\overline{U}_{i+1}(\tilde{S}, Z, \mathcal{M})$ is a convex function with respect to $\tilde{S}_{i+1} \in [dS_i, uS_i]$. Then, with the same calculations that we use for the case $n-1$, we can prove that $\overline{U}_i(S, Z, \mathcal{M})$ is convex and generated by (5.2) for all $S \in \mathcal{S}$ such that $M(S) = n$. Reasoning by induction over $i = n - M(S)$, (5.2) is generalized for all $S \in \mathcal{S}$. \square

We now obtain some simplifications that apply to arbitrage 0-neutral nodes, towards this end, we refine Definition 7.

Definition 13. Consider a discrete market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$, and a node (S, i) .

(1) We call (S, i) a positive arbitrage node if

$$\sup_{\hat{S} \in \mathcal{S}_{(S,i)}} (\hat{S}_{i+1} - S_i) > 0 \text{ and } \inf_{\hat{S} \in \mathcal{S}_{(S,i)}} (\hat{S}_i - S_i) = 0.$$

(2) We call (S, i) a negative arbitrage node if

$$\sup_{\hat{S} \in \mathcal{S}_{(S,i)}} (\hat{S}_{i+1} - S_i) = 0 \text{ and } \inf_{\hat{S} \in \mathcal{S}_{(S,i)}} (\hat{S}_i - S_i) < 0.$$

Observe that in a negative arbitrage node $\mathcal{S}_{(S,i)} = \mathcal{S}_{(S,i)}^-$, while in a positive arbitrage node

$$\mathcal{S}_{(S,i)}^- = \{S' \in \mathcal{S}_{(S,i)} : S'_{i+1} = S_i\} \equiv \mathcal{S}_{(S,i)}^-.$$

Corollary 3. Let $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ be an n -bounded discrete market, and consider the hypothesis of Theorem 6 item (1). For any node (S, i) , $0 \leq i < M(S)$, which is either a negative arbitrage node, or a positive arbitrage node, and $\mathcal{S}_{(S,i)}^-$ is nonempty, it holds that

$$\overline{U}_i(S, Z, \mathcal{M}) = \sup_{S' \in \mathcal{S}_{(S,i)}^-} \overline{U}_{i+1}(S', Z, \mathcal{M}).$$

Proof. Assume (S, i) is a positive arbitrage node and $S' \in \mathcal{S}_{(S,i)}^- = \mathcal{S}_{(S,i)}^-$. It follows that for any $S^+ \in \mathcal{S}_{(S,i)}^+$

$$\overline{U}_{i+1}(S', Z, \mathcal{M}) = \overline{U}_{i+1}(S', Z, \mathcal{M}) - u_{(S^+, S')} (S'_{i+1} - S_i) = \overline{U}_{i+1}(S^+, Z, \mathcal{M}) - u_{(S^+, S')} (S_{i+1}^+ - S_i).$$

Thus $L_i(S, Z, \mathcal{M}) = \sup \{ \overline{U}_{i+1}(S', Z, \mathcal{M}) : S' \in \mathcal{S}_{(S,i)}^- \}$ and the result follows from Theorem 6.

For a negative arbitrage node, the proof is similar and making use of Theorem 6 with the alternative definitions for $\mathcal{S}_{(S,i)}^-$, $\mathcal{S}_{(S,i)}^+$. \square

6. EXAMPLES: TRAJECTORY SETS VIA ANOTHER SOURCE OF UNCERTAINTY

This section provides examples of trajectory sets defined by means of an additional source of uncertainty (denoted by W) besides the stock. Section 2.2 explained how the formalism accommodates such an augmented setting. A general class of models and a discretized version of them are developed as well as a concrete example based on sampled quadratic variation.

6.1. Construction of Trajectory Sets From Augmented Data. The models to be described here were partially introduced in [9]. We refer to that paper for an extended discussion on how to interpret the models, for this reason, explanations below are kept brief. One should not get the wrong impression that these are the only possible models supported by the formalism, on the contrary, the approach allows for trajectory sets that could be constructed from historical data, random samples from large collection of trajectories, etc. We refer to Section 10 from [9] where trajectory sets are constructed by sampling paths of continuous time martingales.

The general principle guiding the construction is to isolate an observable quantity (representing a variable of interest) and proceed to define a trajectory space by imposing constraints relating the trajectories and a free variable representing this observable. In some cases, this process allows to impose natural constraints that follow from the discrete nature of the financial transactions. For simplicity W is chosen to be one dimensional and in applications is meant to be associated to the values taken by an observable quantity which unfolds along the stock *chart* $x(t)$. This latter quantity could unfold in continuous time and its future values be influenced by a source of uncertainty encoded in W . Thus, we use the augmented formalism introduced in Section 2.2 and so, trajectories are sequences of tuples $\mathbf{S} = \{(S_i, W_i)\}_{i \geq 0}$.

There is no essential result in our paper that requires $S_i \geq 0$, so there is no need to use the exponential function in the definition $S_i = e^{x(t_i)}$ but, doing so makes it easier to connect with the usual geometric stochastic models as well as with [7]. Notice that when using $S_i = e^{x(t_i)}$ one should use the word *charts* for the data $e^{x(t)}$, instead of $x(t)$ as we do, but we allow ourselves some abuse of language at this point.

The definition below assumes given: $w_0 = 0$, s_0 and $c, d > 0$ real numbers, $\Sigma_i \subseteq \mathbb{R}$ and sets $Q, \Omega_i \subset (0, \infty)$.

Definition 14. $\mathcal{S}^{\mathcal{W}}(s_0, d, c, Q)$ will denote a subset of $\mathcal{S}_{\infty}^{\mathcal{W}}(s_0, w_0)$ (this last set as in (2.6)) so that $\mathbf{S} \in \mathcal{S}^{\mathcal{W}}(s_0, d, c, Q)$ satisfies $S_i \in \Sigma_i$, $W_i \in \Omega_i$ and:

- (1) $|\log S_{i+1} - \log S_i| \leq d$ for all $i \geq 0$,
- (2) $0 < W_{i+1} - W_i \leq c$ for all $i \geq 0$.

Moreover, there exists at least one i^* such that

- (3) $W_{i^*} \in Q$.

Associated discrete markets $\mathcal{S}^{\mathcal{W}}(s_0, c, d, Q) \times \mathcal{H}$ are required to satisfy: $H \in \mathcal{H}$ then $W_{N_H(\mathbf{S})} \in Q$.

Remark 6. As already mentioned, the condition $|\log S_{i+1} - \log S_i| \leq d$ could equally be replaced by $|S_{i+1} - S_i| \leq d$ (of course with an appropriately chosen value for d).

We emphasize that $\mathcal{S}^{\mathcal{W}}(s_0, d, c, Q)$, as characterized above, does not need to be, in general, the set of *all* trajectories \mathbf{S} satisfying the listed constraints in Definition (14). For $\mathbf{S} \in \mathcal{S}^{\mathcal{W}}(s_0, d, c, Q)$ there could be multiple indexes i^* satisfying (3) above.

Sets of type $\mathcal{S}^{\mathcal{W}}(s_0, c, d, Q)$ can be used for modeling the unfolding of a data chart $x(t_i)$ by mapping $\{(e^{x(t_i)}, F(x, t_0, t_i))\}$, one index i at a time (i.e. as the chart unfolds), to its closest path $\{(S_i, W_i)\}_{i \geq 0}$. Here $F(x, t_0, t_i)$ is an observable quantity that changes as the path unfolds, say the volume of transactions, quadratic variation, etc. In the context of an option contract expiring at time T , $S_{N_H(\mathbf{s})}$ will be a possible value being taken by $e^{x(T)}$. In the case when W_i represents the quadratic variation of the chart $x(t)$ up to time t_i , the constraints defining $\mathcal{S}^{\mathcal{W}}(s_0, d, c, Q)$ can be interpreted as imposing constraints on the consumed quadratic variation and on the absolute value of the change on chart values, both in between consecutive trading instances. The condition $W_{N_H(\mathbf{s})} \in Q$ means we deal with trajectories whose total quadratic variation in the interval $[0, T]$ belongs to the a-priori given subset Q . It should be clear that one can use other algebraic combinations, such as $\sqrt{(W_{i+1} - W_i) |\log S_{i+1} - \log S_i|}$ and impose constraints on such expressions.

The trajectory set introduced in [7] can be recovered as a special case of Definition 14 by taking $Q = \{v_0\}$ and defining the set of *all* trajectories $\{(S_i, W_i)\}$ such that item (1) from Definition 14 holds and defining

$$(6.1) \quad W_i = \sum_{k=0}^{i-1} (\log S_{k+1} - \log S_k)^2,$$

moreover one requires that there exists i^* satisfying $W_{i^*} = v_0$. Therefore $W_{i+1} - W_i = (\log S_{i+1} - \log S_i)^2$ and, by taking $c \equiv d^2$, the constraint $W_{i+1} - W_i \leq c$ in Definition 14 holds. Moreover, as W_i depends on S_k , $0 \leq k \leq i$, there is no need to work with tuples (S_i, W_i) in this case.

The introduction of W_i as an independent variable, in particular not defined by (6.1), allows to widen the scope of applicability of the model given by Definition 14, in particular, now W_i can represent any variable of interest such as number or volume of transactions, time, quadratic variation, etc. Moreover, not imposing (6.1) allows to incorporate arbitrage 0-neutral nodes (see Definition 7). This is so as we can have $(\log S_{i+1} - \log S_i) = 0$ while having $0 < W_{i+1} - W_i$.

The following section describes an instance of the above introduced class of trajectory sets.

6.2. Quadratic Variation Case. A specific trajectory set of the type $\mathcal{S}^{\mathcal{W}}(s_0, c, d, Q)$ is introduced here; it presents a concrete example of how one may map chart values $x(t)$, on an interval $[0, T]$, to a set of trajectories $\mathbf{S} \in \mathcal{S}^{\mathcal{W}}$, the latter to be defined.

Market charts belong to a subset of all functions $x : [0, \infty) \rightarrow \{k\delta_0 : k \in \mathbb{N}\}$ where δ_0 is the minimum possible change of a chart value and $x(0) = S_0$ for all x . We do not model the admissible charts x directly but obtain necessary conditions satisfied by their samples in an operational setting involving sampled quadratic variation defined by:

$$(6.2) \quad w(x, t) \equiv \sum_{r_{l+1} \leq t} (x(r_{l+1}) - x(r_l))^2,$$

and the assumption that, for each x , the dynamic sampling times $\{r_l\}, 0 = r_0 < r_1 < \dots$, obey

$$(6.3) \quad \delta_0 \leq |x(r_{l+1}) - x(r_l)| \leq \eta \quad \forall l, \text{ moreover, there exists } M_1 \text{ so that } r_{M_1} = T.$$

where η is a given constant. We call the right hand side of the above inequality a *jump* constraint. In practice one may construct the samples r_l recursively by, given r_l , letting r_{l+1} to be the smallest real number satisfying (6.3).

Therefore, we work in a setting where sampled quadratic variation for an unfolding chart x is measured along times $\{r_l\}$, a set that will be x -dependent. For convenience each change $r_l \rightarrow r_{l+1}$ will be referred as a *time-tick*.

Portfolio rebalances take place along times $\mathbf{t} = \{0 \leq t_0 < t_1 < \dots < t_{M_2} = T\} \subseteq \{r_l\}$, with $M_2 = M_2(\mathbf{t})$. We do not necessarily assume consecutive rebalancing, i.e. if $t_i = r_{l_i}$, in general $t_{i+1} \geq r_{l_i+1}$; notice that $r_{M_1} = t_{M_2}$ and $M_2 \leq M_1$.

The basic operational and modeling hypothesis are given by the definition (6.2) and the assumption (6.3). We will associate $x(t_i)$ to S_i and may write $x(t_i) \rightarrow S_i$.

The charts $x(t)$ take discrete values $x(r_l) = S_0 + n_l \delta_0$ with $n_l \in \mathbb{Z}$ so, we will set $S_i = S_0 + k_i \delta_0$. We define the augmented trajectory coordinates $W_i = j_i \delta_0^2$ and associate $w(x, t_i) \rightarrow W_i$.

Regarding the set \mathcal{Q} , one can proceed in two alternative ways; first by setting \mathcal{Q} to be a set containing all possible values taken by

$$(6.4) \quad w(x, T) = \sum_{l+1 \leq M_1} (x(r_{l+1}) - x(r_l))^2, \text{ where } x \text{ ranges over the set of charts.}$$

Secondly, by prescribing a smaller set \mathcal{Q} and explicitly assuming the values attained by (6.4) belong to such a set \mathcal{Q} for all charts x . In either case $\mathcal{Q} \subseteq \{n \delta_0^2 : n \in \mathbb{N}\}$.

Assume also the existence of an option Z in the interval $[0, T]$. We will be making assumptions on investors who re-balance their portfolios along such x and at times $t_i \in \{r_l\}$ and that are involved in trading to approximate the option's payoff maturing at T . Investor's $H \in \mathcal{H}$ could also trade to take advantage of any perceived miss-price in the option, in particular, investors will consider the expiration time T while setting up their portfolios, this will be reflected by a constraint on the quadratic variation of the form $W_{N_H}(\mathbf{s}) \in \mathcal{Q}$. That is, we associate $x(T) \rightarrow S_{N_H}(\mathbf{s})$.

Given a node (S_i, W_i) we define possible values (S_{i+1}, W_{i+1}) by providing a set of possible pair of integers (k_{i+1}, j_{i+1}) . This set of possible (pairs) of integers is motivated by chart changes under which investors may decide to re-balance their portfolios. In this way, the model will show the impact on minmax prices due to the operational setup.

Assuming a trajectory has moved an arbitrary amount $\Delta_i S = S_{i+1} - S_i \equiv (k_{i+1} - k_i) \delta_0$ we will obtain a relationship with a corresponding change in $\Delta_i W \equiv (j_{i+1} - j_i) \delta_0^2$. The stock having moved from $S_i = x(t_i)$ to $S_{i+1} = x(t_{i+1})$, it follows that there exist l_1, L such that $t_i = r_{l_1}$ and $t_{i+1} = r_L$ with $L = l_1 + m$, $m \in \mathbb{N}$. Therefore

$$(6.5) \quad \Delta_i S = S_{i+1} - S_i = x(r_L) - x(r_{l_1}) = \sum_{j=0}^{m-1} (x(r_{l_1+j+1}) - x(r_{l_1+j})).$$

Since $\{r_{l_1+j}\}_{0 \leq j < m}$ are consecutive elements in our fine sampling, we also have

$$w(x, t_{i+1}) - w(x, t_i) = \sum_{j=0}^{m-1} (x(r_{l_1+j+1}) - x(r_{l_1+j}))^2.$$

We can write $x(r_{l_1+j+1}) - x(r_{l_1+j}) = n_{l_1+j+1} - n_{l_1+j} \delta_0 = m_j \delta_0$ with $m_j \in \mathbb{Z}$; this fact and (6.5) imply

$$(6.6) \quad w(x, t_{i+1}) - w(x, t_i) = \delta_0^2 \sum_{j=0}^{m-1} m_j^2 = \delta_0 \sum_{j=0}^{m-1} |m_j| \delta_0 \geq \delta_0 \left| \sum_{j=0}^{m-1} m_j \delta_0 \right| = \delta_0 |\Delta_i S|.$$

Using the association $[w(x, t_{i+1}) - w(x, t_i)] \rightarrow \Delta_i W$ we set the following constraint for our trajectory model:

$$(6.7) \quad |\Delta_i S| \delta_0 \leq \Delta_i W.$$

Or, equivalently

$$(6.8) \quad |k_{i+1} - k_i| \leq (j_{i+1} - j_i).$$

Notice that (6.8) provides a dynamic relationship between $\Delta_i S$ and $\Delta_i W$.

At first sight a trajectory model given by (6.8) seems misspecified as there is no upper bound for j_{i+1} for a given j_i . This issue can be remedied by assuming Q to be bounded and noticing that this implies $j_{i+1} < J$ where J is the smallest integer such that $J\delta_0^2 \geq \sup Q$. Therefore, considering $j_{i+1} \geq J$ will be irrelevant because necessarily $M_1 < J$ for any possible M_1 .

Next, we add a further operational assumption implying an upper bound for $\Delta_i W$ and a tighter lower bound for $\Delta_i W$ as well. Specifically, we consider those investors that will contemplate rebalancing their portfolios at, or prior than, A time-ticks. A is an integer satisfying $A \geq 1$. Another way to say this, is that these investors do not want to face the risk originating from contemplating no portfolio rebalance during the next A time-ticks following their last portfolio rebalancing. It follows that

$$w(t_{i+1}, x) - w(t_i, x) \leq A \eta^2 \text{ for each } i.$$

Notice that from (6.3) we also obtain $|\Delta_i S| \leq A\eta$. We can also repeat the previous analysis leading to (6.7); specifically, we modify (6.6) as follows:

$$w(x, t_{i+1}) - w(x, t_i) = \delta_0^2 \sum_{j=0}^{m-1} m_j^2 \geq \delta_0^2 \left| \sum_{j=0}^{m-1} \frac{m_j}{\sqrt{m}} \right|^2 = \frac{1}{m} \left| \sum_{j=0}^{m-1} m_j \delta_0 \right|^2 = \frac{\Delta_i S^2}{m} \geq \frac{\Delta_i S^2}{A}.$$

Making use of the above obtained relationships, we define $\mathcal{S}^{\mathcal{W}}$ to be the set of all $\mathbf{S} = \{(S_i, W_i)\}$ so that

$$\max(|S_{i+1} - S_i| \delta_0, \frac{\Delta_i S^2}{A}) \leq (W_{i+1} - W_i) \leq A \eta^2 \text{ and } |\Delta_i S| \leq A\eta.$$

Or, equivalently,

$$\max(|k_{i+1} - k_i|, \frac{|k_{i+1} - k_i|^2}{A}) \leq (j_{i+1} - j_i) \leq \frac{A \eta^2}{\delta_0^2} \text{ and } |k_{i+1} - k_i| \leq \frac{A \eta}{\delta_0}.$$

Moreover, $Q \subseteq \{n\delta_0^2 : n \in \mathbb{N}\}$ is assumed to be such that for each \mathbf{S} as above there exists at least i^* with $W_{i^*} \in Q$. One should also specify the range sets Σ_i and Ω_i (something not essential for the example at this point and so we will skip it). We will specify these quantities in Section 8 along with an explicit Q .

Taking into consideration Remark 6, $\mathcal{S}^{\mathcal{W}}$ then satisfies Definition 14, and could then be denoted by $\mathcal{S}^{\mathcal{W}}(s_0, d = A\eta, c = A \eta^2, Q)$.

Given x and \mathbf{t} , it follows that the chart assignments $x(t_i) \rightarrow S_i$, $w(t_i, x) \rightarrow W_i$ gives $\mathbf{S} = \{(S_i, W_i)\} \in \mathcal{S}^{\mathcal{W}}$ as long as (6.3) holds. For each such \mathbf{S} , an investor $H \in \mathcal{H}$, rebalances along times $\mathbf{t} = \{t_i\}$ with liquidation index $N_H(\mathbf{S}) = M_2(\mathbf{t})$.

7. DISCRETIZATION AND GRID DATA STRUCTURE

7.1. Discretization. A natural discretization leading to an implementation of sets of type $\mathcal{S}^{\mathcal{W}}(s_0, d, c, Q)$ is obtained by introducing real numbers $\delta, \beta > 0$. The coordinates S_i are restricted to belong to the sets

$$\Sigma_i = \Sigma(\delta) \equiv \{s_0 e^{k\delta}, k \in \mathbb{Z}\}$$

and the coordinates W_i to

$$\Omega_i \equiv \Omega(\beta) = \{j\beta^2, j \in \mathbb{N}\},$$

we will also take $Q(\beta) \subseteq \Omega(\beta)$.

The fact that the variable W_i is not directly related to the samples S_i , in particular, we are not assuming (6.1), makes natural to have two distinct discretization parameters, namely δ and β . The parameter δ clearly provides a natural discretization of the chart exponentials.

For implementation purposes we need a finite version of the above discrete space. Towards this end we will take, for convenience, $p = \frac{d}{\delta}$ and $q = \frac{c}{\beta^2}$ two fixed integers. For given $N_1, N_2 \in \mathbb{N}$ define

$$\Sigma_i \equiv \Sigma(\delta, N_1) = \{s_0 e^{k\delta}, k \in \{-N_1, -N_1 + 1, \dots, N_1\}\}, \quad \Omega_i \equiv \Omega(\beta, N_2) = \{j\beta^2, 0 \leq j \leq N_2\},$$

and for a finite m -tuple of positive integers $\Lambda = (n_1, \dots, n_m)$, define

$$Q \equiv Q(\beta, \Lambda) = \{n_k \beta^2 : 1 \leq k \leq m\}.$$

Given the above definitions we let $\mathcal{S}^{\mathcal{W}}(s_0, p, q, \Lambda, N_1, N_2, \delta, \beta)$ to denote a set $\mathcal{S}^{\mathcal{W}}(s_0, d, c, Q)$ as in Definition 14. Notice that, necessarily, any possible value of i^* satisfies $i^* \leq N_2$, so, by Definition 9, the corresponding markets will be N_2 -bounded. In any case, it is clear that sets of type $\mathcal{S}^{\mathcal{W}}(s_0, p, q, \Lambda, N_1, N_2, \delta, \beta)$ will have finite cardinality. Figure 1 from [9] displays random trajectories for the case when $\mathcal{S}^{\mathcal{W}}(s_0, p, q, \Lambda, N_1, N_2, \delta, \beta)$ is defined as the set of *all* trajectories satisfying the requirements in Definition 14. Section 7.2 gives more details on how to build these sets as well as their properties.

Local Behavior. Specific instances of the sets $\mathcal{S}^{\mathcal{W}}(s_0, d, c, Q)$ or their finite versions $\mathcal{S}^{\mathcal{W}}(s_0, p, q, \Lambda, N_1, N_2, \delta, \beta)$ will in fact impose further constraints on admissible trajectories. Once these further specifications are established, the resulting trajectory sets are defined in a combinatorial way (i.e. by allowing all possible $\{(S_i, W_i)\}$ satisfying the constraints). One such example was described in Section 6.2 and further examples are in Section 8. This way of defining trajectory sets will make it easy to check if the local properties of 0-neutral or up-down are satisfied. This is so because our constraints are given locally (i.e. at each node) and the combinatorial definitions will allow trajectories to move up or down. Next, as examples, we provide some general arguments on how to argue for the validity of these local properties. Assume that the sets Σ_i in the trajectory space $\mathcal{S}^{\mathcal{W}}(s_0, d, c, Q)$ of Definition 14 do not attain minimum nor maximum and fix a node (S_i, W_i) of a trajectory \mathbf{S} . Clearly, there exists the possibility of choosing trajectories $\tilde{\mathbf{S}}, \hat{\mathbf{S}} \in \mathcal{S}^{\mathcal{W}}_{(\mathbf{S}, i)}$ such that $\tilde{S}_{i+1} > S_i$, and $\hat{S}_{i+1} < S_i$ respectively, so any node is up-down, and in that case the market results locally arbitrage-free, see Definition 7.

Consider now $\mathcal{S}^{\mathcal{W}}(s_0, p, q, \Lambda, N_1, N_2, \delta, \beta)$, in this case we can assume that $N_2 \beta^2 \in Q$. First note, as previously indicated, that for any trajectory $\mathbf{S} = \{(S_i, W_i)\}_{i \geq 0}$, since $W_{i^*} \in Q \subset \Omega(\beta, N_2)$, then $i^* \leq N_2$. Taking into account the constraint $d \geq |\log S_{i+1} - \log S_i| = |k_{i+1} - k_i| \delta$, the largest

value that S_i can attain corresponds to the value S_{N_2} and, in that case, $S_{N_2} = s_0 e^{N_2 p \delta}$, which shows $N_1 \leq p N_2$.

In the case that $N_1 \leq (N_2 - 1)p$, could exist trajectories containing the node $(S_{i'} = s_0 e^{N_1 \delta}, i' \beta^2)$ with $i' \leq N_2 - 1$ (for example, by choosing $k_i = ip$ for $i > 0$, when $i \leq \frac{N_1}{p}$ and at most $k_{N_2-1} = N_1$). Such trajectory satisfies $W_i \leq (N_2 - 1)\beta^2$ and so, one more step is available. Moreover, given that for any trajectory $\tilde{\mathbf{S}} \in \mathcal{S}_{(\mathbf{S}, i')}^{\mathcal{W}}$ it follows that $\tilde{S}_{i'+1} \leq \tilde{S}_{i'} = S_{i'} = s_0 e^{i' p \delta}$, (\mathbf{S}, i') is an arbitrage node. These nodes present arbitrage opportunities.

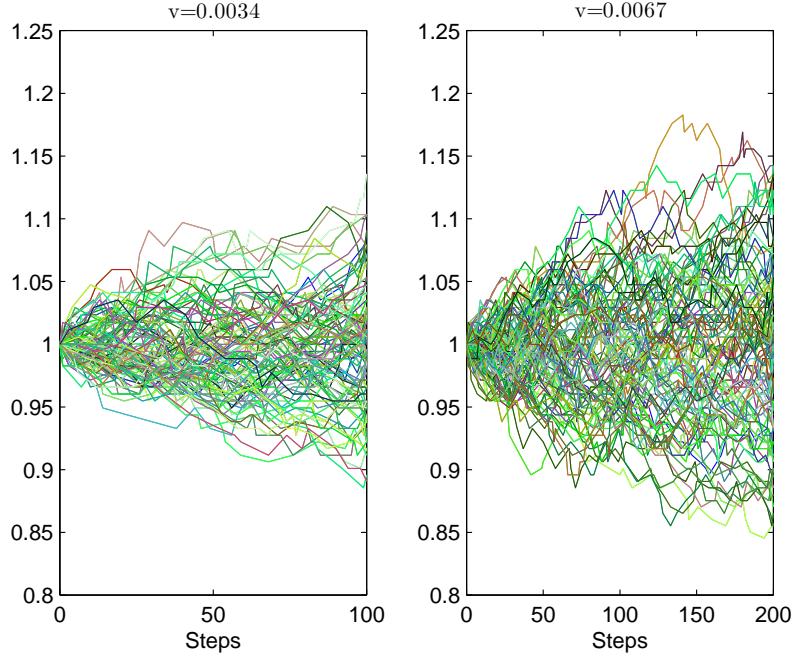


FIGURE 1. Trajectory sets with different quadratic variation for $s_0 = 1$, $w_0 = 0$, $\delta = \beta = 0.0058$, $N_1 = 300$, $N_2 = 200$, $p=3$, and $c = d^2$.

Figure 1 shows 100 random trajectories, in each display. The sets are shown in different displays for convenience but they belong to the same trajectory set, with $\Lambda = (n_1, n_2) = (100, 200)$. The case of $v = 0.0034 = 100 \beta^2$ correspond to trajectories with $i^* \leq 100$, while the value $v = 0.0067 = 200 \beta^2$ corresponds to trajectories with $i^* \leq 200$.

7.2. Computational Grid. Here we are going to introduce a grid of pairs of integer numbers Γ , which will be used to represent the trajectories of a discrete finite market $\mathcal{M} = \mathcal{S}^{\mathcal{W}} \times \mathcal{H}$ with $\mathcal{S}^{\mathcal{W}}$ a trajectory set of the type $\mathcal{S}^{\mathcal{W}}(s_0, p, q, \Lambda, N_1, N_2, \beta, \delta)$ described in the previous section. The purpose of the grid Γ is to implement the algorithm given by Theorem 6 in order to evaluate the dynamic bounds $\bar{U}_i(\mathbf{S}, \mathbf{Z}, \mathcal{M})_{i \geq 0}$ with $\mathbf{S} \in \mathcal{S}^{\mathcal{W}}$. Consequently, under appropriate conditions, we will obtain also the global bound $\bar{V}_0(s_0, \mathbf{Z}, \mathcal{M})$. In the rest of the section \mathbf{S} will stand for a trajectory $\mathbf{S} \equiv \{(S_i, W_i)\}_{i \geq 0} \in \mathcal{S}^{\mathcal{W}}$.

Given the discretization parameters β, δ positive real numbers, $p = \frac{d}{\delta}, q = \frac{c}{\beta^2}$ (it gives d, c) and $N_1, N_2 \in \mathbb{N}$, call *trajectory grid* to

$$\Gamma = \{(k, j) : |k| \leq N_1, 0 \leq j \leq N_2, -pj \leq k \leq pj\}.$$

For integers $1 \leq n_1 < n_2 < \dots < n_m$, let $Q = \{n_j \beta^2 : 1 \leq j \leq m\}$ be the set to get the required restriction on $W_{N_H(\mathbf{S})}$, observe that is natural that $N_2 \beta^2 \in Q$.

For any $i \geq 0$, each node (S_i, W_i) can be represented by a vertex $(k_i, j_i) \in \Gamma$, such that

$$(7.1) \quad S_i = s_0 e^{k_i \delta} \quad \text{and} \quad W_i = j_i \beta^2.$$

In Section 7.1 it has shown that it is enough that $N_1 \leq p N_2$. Also observe that the restriction $-pj \leq k \leq pj$ obey to the one on \mathcal{S}^W given by the constraint $d \geq |\log S_{i+1} - \log S_i| = \delta |k_{i+1} - k_i|$, which implies $-p + k_i \leq k_{i+1} \leq k_i + p$.

On the other hand, any sequence $(k_i, j_i)_{i \geq 0}$, with $|k_{i+1} - k_i| \leq p, 0 < j_{i+1} - j_i \leq q$ for $i \geq 0$ and $j_{i^*} \beta^2 \in Q$ for some $i^* \leq N_2$, corresponds by the same association (7.1), to a trajectory $\mathbf{S} \in \mathcal{S}^W$.

7.3. Computation of Prices in the Grid. The trajectory grid Γ presented above will be used to compute the dynamic bounds $\bar{U}_i(\mathbf{S}, \mathbf{Z}, \mathcal{M})$ for a discrete n -bounded finite market $\mathcal{M} = \mathcal{S}^W \times \mathcal{H}$, using The Convex Hull Theorem (Theorem 6). This can be done for an European option given by a function Z defined on \mathcal{S}^W . For simplicity this option is assumed independent of the trajectory history, namely $Z(\mathbf{S}) = Z^f(S_{N_H(\mathbf{S})})$ for a real variable function Z^f (notation introduced in Corollary 2). This condition on Z allows to compute the dynamic bounds on the vertices of Γ as follows. For simplicity we will use the notation $s_k = s_0 e^{k \delta}$. Also assume that the set of portfolios \mathcal{H} is composed for sequences $H = \{H_i\}_{i \geq 0}$ including any function from \mathcal{S}^W to \mathbb{R} , non anticipative with respect to i , could be one of the $H_i, i \geq 0$, thus \mathcal{H} is FULL.

First, we consider $Q = \{N_2 \beta^2\}$; the dynamic bounds $\bar{U}_i(\mathbf{S}, Z, \mathcal{M})$ for $0 \leq i \leq N_2$, can be associated to the vertices of Γ . Indeed since $W_{N_H(\mathbf{S})} = N_2 \beta^2$ the node $(S_{N_H(\mathbf{S})}, W_{N_H(\mathbf{S})})$ corresponds by (7.1) to some $(k_N, N_2) \in \Gamma$ ($N \equiv N_H(\mathbf{S})$), then $\bar{U}_N(\mathbf{S}, Z, \mathcal{M}) = Z^f(s_{k_N})$. Moreover whenever the trajectory $\tilde{\mathbf{S}}$ has a node $(s_{k_N}, N_2 \beta^2)$, will have $\bar{U}_N(\tilde{\mathbf{S}}, Z, \mathcal{M}) = Z^f(s_{k_N})$. Now observe that if (k_{i^0}, j_{i^0}) correspond to a node (S_{i^0}, W_{i^0}) , $\bar{U}_{i^0}(\mathbf{S}, Z, \mathcal{M})$ depends, by Definition 10 and Theorem 6, only on $\bar{U}_{i^0+1}(\tilde{\mathbf{S}}, Z, \mathcal{M})$, \tilde{S}_{i^0+1} and S_{i^0} , where $\tilde{\mathbf{S}} \in \mathcal{S}_{(\mathbf{S}, i^0)}^W$. Then, those quantities are associated to the vertices $(k, j) \in \Gamma$ with

$$(7.2) \quad -p \leq k - k_{i^0} \leq p, \quad \text{and} \quad 0 < j - j_{i^0} \leq q.$$

Thus $\bar{U}_{i^0}(\mathbf{S}, Z, \mathcal{M})$ can be associated with the vertex (k_{i^0}, j_{i^0}) , via a function \bar{U} with domain Γ . Actually, we are going to define \bar{U} so that $\bar{U}(k_{i^0}, j_{i^0}) = \bar{U}_{i^0}(\mathbf{S}, Z, \mathcal{M})$. Vertices $(k, j) \in \Gamma$ verifying (7.2) are called *reachable* from (k_{i^0}, j_{i^0}) .

For each vertex $(k, j) \in \Gamma$ we define \bar{U} by the following procedure.

Since any vertex $(k, N_2) \in \Gamma$ corresponds to a (S_N, W_N) node of a trajectory in \mathcal{S}^W , with $W_N \in Q$, define

$$\bar{U}(k, N_2) = Z^f(s_k), \quad \text{for any } k : |k| \leq N_1.$$

Now assume, for fixed $j < N_2$, $\bar{U}(k^*, j^*)$ was defined for any $j^* : j < j^* \leq N_2$, and any $k^* : |k^*| \leq p j^*$. For fixed $(k, j) \in \Gamma$ and any pairs $(k^+, j^+), (k^-, j^-)$ verifying

$$(7.3) \quad -p \leq k^- - k \leq 0 < k^+ - k \leq p, \quad 0 < j^+ - j \leq q \quad \text{and} \quad 0 < j^- - j \leq q,$$

set

$$u_{+,-} \equiv \frac{\overline{U}(k^+, j^+) - \overline{U}(k^-, j^-)}{s_{k^+} - s_{k^-}}.$$

Being $\mathbf{S} \in \mathcal{S}^{\mathcal{W}}$ a trajectory such that (S_k, W_j) corresponds by (7.1) to (k, j) , it is important to notice the following: Since the pairs (k^+, j^+) , (k^-, j^-) verifying (7.3) are reachable from (k, j) , if $\mathbf{S}^+, \mathbf{S}^- \in \mathcal{S}_{(\mathbf{S}, k)}^{\mathcal{W}}$ verify that $(S_{k^+}^+, W_{j^+}^+)$ and $(S_{k^-}^-, W_{j^-}^-)$ corresponds respectively to (k^+, j^+) , (k^-, j^-) , then $\mathbf{S}^+ \in \mathcal{S}_{(\mathbf{S}, k)}^{\mathcal{W}^+}$ and $\mathbf{S}^- \in \mathcal{S}_{(\mathbf{S}, k)}^{\mathcal{W}^-}$. Consequently (5.1) is applicable and $\overline{U}(k, j) \equiv C(k, j)$ is defined according to it, by

$$(7.4) \quad C(k, j) = \sup \{ \overline{U}(k^+, j^+) - u_{+,-}(s_{k^+} - s_k) \}, \quad \text{for } 0 \leq j < N_2 \text{ and } |k| \leq pj,$$

where the supremum is taken over the pairs $(k^+, j^+), (k^-, j^-)$ verifying (7.3).

Therefore, the above recursive procedure allows to obtain $\overline{U}(0, 0) = \overline{U}_0(s_0, Z, \mathcal{M}) = \overline{V}(s_0, Z, \mathcal{M})$, since the hypothesis of Theorem 3 are satisfied.

We now extend the procedure to a set Q given by an strictly increasing m -tuple $\Lambda = (n_1, \dots, n_m)$ with $n_m = N_2$. We start from column $j = n_m$. Again, if a trajectory contains the value represented by the node (k, n_m) , then define $\overline{U}(k, n_m) = Z^f(s_k)$.

For a vertex $(k, j) \in \Gamma$ with $n_{m-1} < j < n_m$, and $k \in [-pj, pj]$, $\overline{U}(k, j)$ is given by (7.4).

The vertices on the column n_{m-1} in Γ , correspond by (7.1) to trajectories \mathbf{S} that could have $N_H(\mathbf{S}) = n_{m-1}$ at that node, it is $W_N = n_{m-1}\beta^2$, or continue to get $W_N = n_m\beta^2$. Thus for $k^* \in [-pn_{m-1}, pn_{m-1}]$, $\overline{U}(k^*, n_{m-1})$ should take the value $Z^f(s_{k^*})$ in the first case, while in the second case its value at that vertex, should be given by (7.4). Both situations must be contemplated to compute $\overline{U}(k, j)$ for $j < n_{m-1}$, by mean of (7.4), when any of the vertices (k^*, n_{m-1}) is reachable from (k, j) . Then, observing that the maximum of these two values is the one which contributes to (5.1), in the referred computation, and Theorem 6, we have

$$\overline{U}(k^*, n_{m-1}) = \max\{Z^f(s_{k^*}), C(k^*, n_{m-1})\}.$$

Following the same considerations, $\overline{U}(k, j) \equiv C(k, j)$ is defined by (7.4) for all $n_r < j < n_{r+1}$ with $1 \leq r < m-1$ and $k \in [-pj, pj]$. For $j = n_r$ with $1 \leq r < m-1$ and $k \in [-pn_r, pn_r]$

$$\overline{U}(k, n_r) = \max\{Z(s_k), C(k, n_r)\}$$

where $C(k, n_r)$ is given by (7.4).

To summarize, $\overline{U}(k, j)$ for $0 \leq j \leq N_2$ and $k \in [-pj, pj]$ is given by

$$\overline{U}(k, j) = \begin{cases} Z^f(s_k) & \text{if } j = n_m \\ \max\{Z^f(s_k), C(k, j)\} & \text{if } j = n_1, n_2, \dots, n_{m-1} \\ C(k, j) & \text{in the other case,} \end{cases}$$

where $C(k, j)$ is given by (7.4). With this recursive procedure we can calculate the value of $\overline{U}(0, 0) = \overline{V}(s_0, Z, \mathcal{M})$. In Section 8 we present a numerical analysis of the behavior of $\overline{U}(0, 0)$ calculated using the above algorithm. Figure 2 shows $\overline{U}(0, 0)$ for different jump size and different N_2 , Figure

5 displays the bounds price as a function of s_0 for the same values of p , and Figure 6 displays the arbitrage 0-neutral effect on the bounds for $p = 1$.

Recalling that $\underline{U}_i(\mathbf{S}, Z, \mathcal{M}) = -\overline{U}_i(\mathbf{S}, -Z, \mathcal{M})$, the lower dynamic bounds $\underline{U}_i(\cdot)$ are computed by a similar procedure.

8. NUMERICAL RESULTS

This section provides numerical output illustrating some characteristics of two specific trajectory sets that belong to the general class of models described in Section 7.1. The output shows the behavior of the upper and lower bound prices for call and put options, both as a function of N_2 and the parameter p . Comparisons with the model introduced in [7] and the Black-Scholes model (B&S) are also provided as well as contrasts with the Merton Bounds [12]. The effect of the presence of 0-neutral arbitrage nodes is also displayed. Finally, some superhedging and underhedging approximations and the effect of variable volatility are presented.

We leave to another opportunity to provide numerical results based on market data, some older, very preliminary, results of this nature can be found in [13].

As already indicated, a particular trajectory set, member of the general class introduced in Definition 14, satisfying condition (6.1) was introduced in [7] and the associated model was compared against Black & Scholes. The book [14] contains an extended discussion of that model, which we will refer as the BJ&N model.

We compute the minmax option bound prices using the dynamic programming approach combined with the convex hull result presented in previous sections (Theorem 6). For reasons of space we do not provide details related to the software implementation.

8.1. Comparison with Other Models. This subsection presents numerical output for the general models with trajectory sets $\mathcal{S}^{\mathcal{W}}(s_0, p, q, \Lambda, N_1, N_2, \delta, \beta)$ that were described in Section 7.1. In this and the following subsection we include the additional restriction that $S_{i+1} - S_i \neq 0$ for all $i \geq 0$ and all trajectories $\mathbf{S} = \{(S_i, W_i)\}_{i \geq 0}$. The purpose of this restriction is twofold, first it provides a fair comparison with BJ&N, as (6.2) excludes this possibility as well, second, it provides an (ad hock) way to exclude a globally constant trajectory which makes the lower bound $\underline{V}(Z)$ to collapse to Merton's bound (as proven in [9, Proposition 5]).

Two example models, labeled \mathcal{M}^a and \mathcal{M}^b , are put forward which are of the type $\mathcal{S}^{\mathcal{W}}(s_0, p, q, \Lambda, N_1, N_2, \delta, \beta)$ and also obey some further constraints. Specific constraints, to each of \mathcal{M}^a and \mathcal{M}^b , are listed below; recall that $d = p\delta$ and $c = q\beta^2$.

Market model \mathcal{M}^a : uses $\beta = \delta$ and the restrictions

$$(8.1) \quad |\log S_{i+1} - \log S_i|^2 \leq W_{i+1} - W_i \leq d^2.$$

Observe that in the case of $p = 1$, it coincides with the BJ&N model (this is so as we are requesting $S_{i+1} \neq S_i$).

Model \mathcal{M}^b : this model was introduced in Section 6.2 and we refer to that section for notation. The original model was described for S but we will now apply it to $\log S$, in particular δ_0 then refers to the minimum change in $\log S$, as this will allow us a reasonable comparison to the other models being implemented. We will take $\beta = \delta = \delta_0$, $\eta = p\eta\delta$, $d = A\eta$, $c = A\eta^2$ and $p = Ap\eta$. Notice that then

$d = p\delta$ and by enforcing $c = q\beta^2 = A\eta^2 = Ap_\eta^2\delta^2$ it follows that $q = Ap_\eta^2 = pp_\eta$. The defining constraints are

$$\max(|\log S_{i+1} - \log S_i|, \frac{|\log S_{i+1} - \log S_i|^2}{A}) \leq (W_{i+1} - W_i) \leq Ap_\eta^2\delta^2 \text{ and}$$

$$|\log S_{i+1} - \log S_i| \leq Ap_\eta\delta.$$

In General we allow all possible values for $N \equiv N_H(\mathbf{S})$ such that $W_N \in Q$. For simplicity, we consider first the case of a single quadratic variation value and take $\Lambda = (N_2)$ in both models. This implies that $Q = \{N_2\beta^2\}$ and so $N_H(\mathbf{S}) \leq N_2$ for all H and \mathbf{S} .

Using the notation from (7.1) we recall that $|k_i| \leq N_1$ and $j_i \leq N_2$, moreover, the above imposed constraints imply

$$\text{model } \mathcal{M}^a : \quad 0 < |k_{i+1} - k_i| \leq p \quad \text{and} \quad |k_{i+1} - k_i|^2 \leq j_{i+1} - j_i \leq p^2.$$

$$\text{model } \mathcal{M}^b : \quad 0 < |k_{i+1} - k_i| \leq Ap_\eta \quad \text{and} \quad \max(|k_{i+1} - k_i|, \frac{|k_{i+1} - k_i|^2}{A}) \leq j_{i+1} - j_i \leq Ap_\eta^2.$$

It then follows that $|k_i| \leq j_i$ for any $i \geq 0$, and then $|k_N| \leq j_N \leq N_2$, where as usual $N = N_H(\mathbf{S})$. Therefore, it is enough to consider $N_1 \leq N_2$. Observe (as in Section 7.1) that if $N_1 < N_2$ nodes $(s_0 e^{\pm N_1 \delta}, N_1 \beta^2)$ are arbitrage nodes, for this reason we will take $N_1 = N_2$. In that case, and recalling the extra restriction $S_{i+1} - S_i \neq 0$, all nodes became of up-down type as per Definition 7.

With the aim to compare with the prices given by the Black & Scholes formula, consider a two-month European call option with strike of $K = 1$ on a stock that pays no dividends, with current price $s_0 = 1$ and the volatility of the stock is taken to be equal to $\sigma = 20\%$ in a year. Define

$$v_0 = \sigma^2 \cdot T = 0.04 \cdot \frac{2}{12} = 0.0067.$$

We recover the trajectory set constructed in [7] by taking $Q = \{v_0\}$ and defining W_i by (6.1). The Black & Scholes price of a Call (or put) option is \$0.0326 when $s_0 = K = 1$ with $\sigma = 0.2$.

The parameters used in the figures that follows are given as a function of N_2 and p with

$$(8.2) \quad \begin{aligned} \text{model } \mathcal{M}^a : \quad & \beta^2 = \frac{v_0}{N_2}, \quad \delta = \beta \quad N_1 = N_2. \\ \text{model } \mathcal{M}^b : \quad & \delta_0^2 = \frac{v_0}{N_2}, \quad A = 2, \quad N_1 = N_2. \end{aligned}$$

Figure 2 shows the behavior of the option price when $p = 2, 3, 5$ with N_2 ranging from 20 to 200 for an European call option for the models BJ&N, \mathcal{M}^a and \mathcal{M}^b with $s_0 = 1 = K$. While Figure 3 displays the behavior of an European put option price in the model \mathcal{M}^b for $s_0 = 1$ and different values of K . When $p > 1$, it seems that we have convergence if N_2 increases but the value of the option does not converge to Black & Scholes price due to jumps.

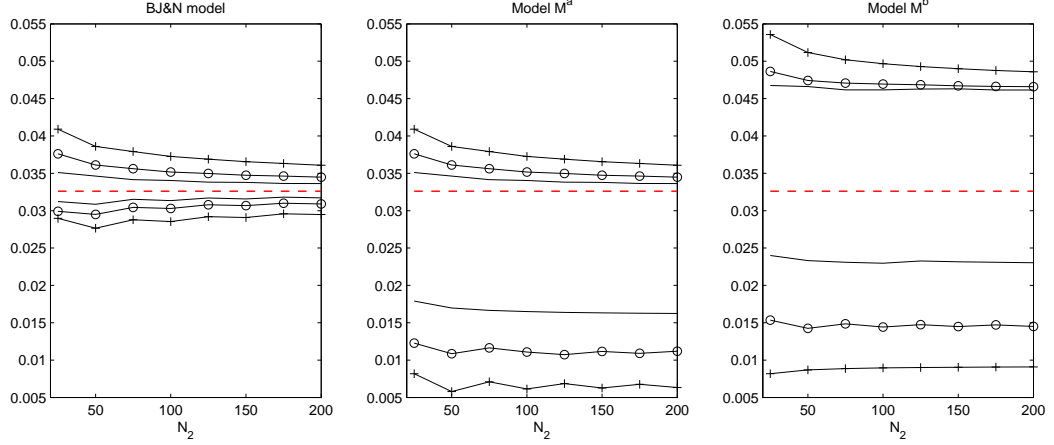


FIGURE 2. Convergence of $\overline{U}(0,0)$ and $\underline{U}(0,0)$ for a European call option as a function of N_2 for $p = 2, 3, 5$ with respect to the B&S price.

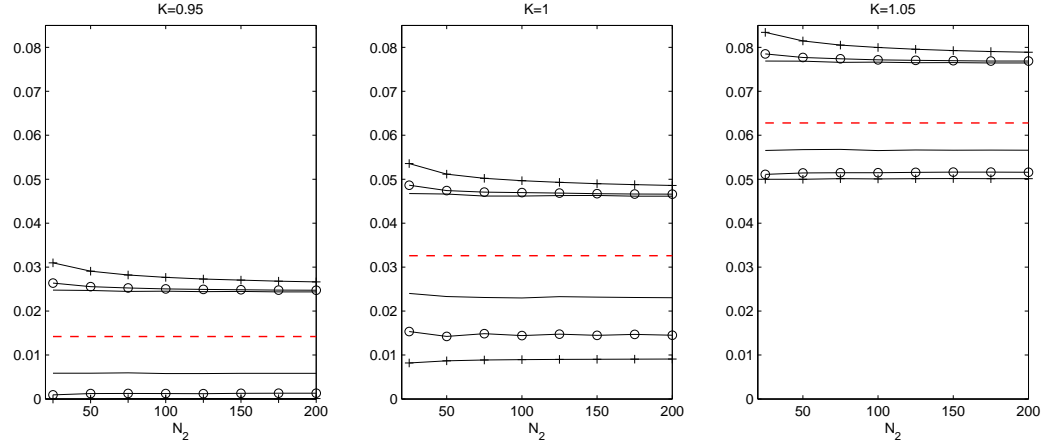


FIGURE 3. Convergence of $\overline{U}(0,0)$ and $\underline{U}(0,0)$ for a European put option as a function of N_2 for $p = 2, 3, 5$ with respect to the B&S price in the \mathcal{M}^b model.

We can see that the value of the option increases when p increase. This fact is due to, if $p < p'$ are two jump size then $\mathcal{S}(p) \subseteq \mathcal{S}(p')$ (where it is used the short notation $\mathcal{S}(p)$ for the set of trajectories corresponding to the parameter p). Therefore, the maximum over the set $\mathcal{S}(p')$ is higher than the maximum over $\mathcal{S}(p)$ when computing the upper bound.

If no jumps are allowed, i.e. $p = 1$, then Figure 4 displays the upper bound price of the option as a function of N_2 under the model \mathcal{M}^a for $s_0 = 1 = K$. The upper minmax bound and the lower minmax bounds are the same. The results for BJ&N, for this case of $p = 1$, are the same as for \mathcal{M}^a . The equality of lower and upper bounds follows immediately from the arguments $\underline{U}(s_0, Z, \mathcal{M}) = -\overline{U}(s_0, -Z, \mathcal{M})$ and $p = 1$.

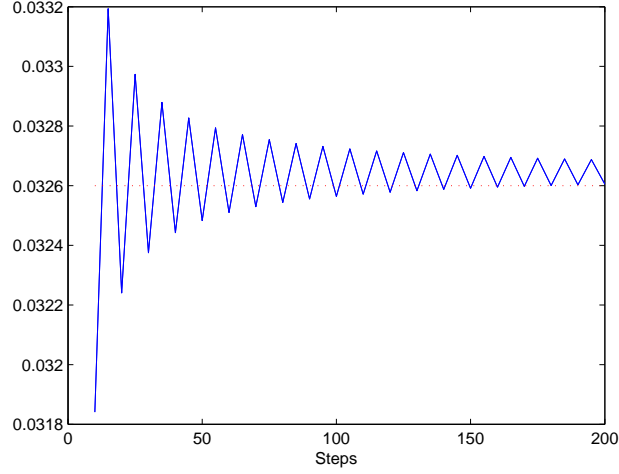


FIGURE 4. The minmax upper and lower bound (coincident in this example), for model \mathcal{M}^a with jump unit $p = 1$, coincide with the BJ&N bounds and approaches the Black-Scholes price as the number of steps N_2 increases.

8.2. Contrasting Against Merton Bounds. Here we plot the minmax bounds as a function of s_0 and in the same set up as above i.e. $K = 1$, $\sigma = 20\%$ per year and the parameters (8.2). We calculate the option values for different starting levels of the stock price s_0 , with $N_2 = 100$ steps in the grid.

Figure 5 displays the minmax bounds for different jump units $p = 1, 3, 5, 7$ for BJ&N and \mathcal{M}^a models, and for $p = 3, 5, 7$ for the \mathcal{M}^b model. In the model \mathcal{M}^b , $p = 1$ is only possible with $A = 1$, and in this case it equals model \mathcal{M}^a . We can see that the price increases when s_0 increases. The minmax bounds are very narrow for larger values of s_0 . Therefore, it seems that the jumps have less of an effect on the bound of the option's prices for the higher stock values.

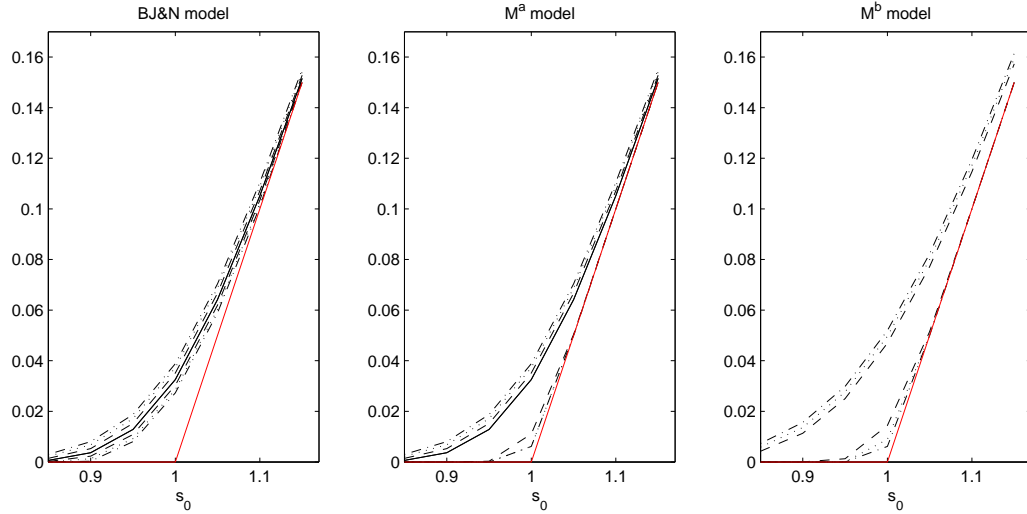


FIGURE 5. Minmax upper and lower bounds price as a function of s_0 for different values of p .

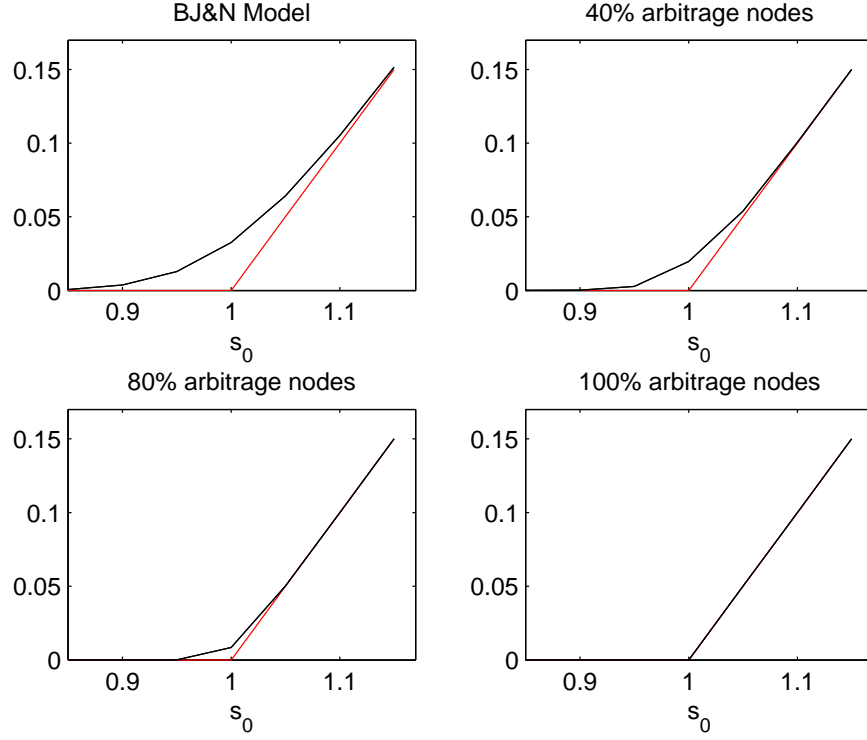


FIGURE 6. Minmax upper and lower bounds price (which are the same in this case) as a function of s_0 for $p = 1$ for the BJ&N model in the presence of arbitrage nodes in relation to the lower Merton bound (red line)

8.3. Effect of Arbitrage Nodes on Minmax Bounds. We calculate the option values for an European call for different starting levels of the stock price s_0 , with $N_2 = 100$ steps in the grid. Some of the nodes of the BJ&N model are changed to be arbitrage 0-neutral (recall that originally all nodes are up-down). Those arbitrage 0-neutral nodes are selected randomly as follows. For any of those selected nodes (k, j) from the BJ&N grid, its reachable vertexes (k', j') (in the BJ&N grid) are changed to:

- If $k \geq 0$, the reachable nodes are (k', j') where

$$-p \leq k' - k \leq 0 \quad \text{and} \quad 0 < j' - j \leq p^2.$$

- If $k < 0$, the reachable nodes are (k', j') where

$$0 \leq k' - k \leq p \quad \text{and} \quad 0 < j' - j \leq p^2.$$

Observe that the modified model has trajectories with $S_{i+1} = S_i$ passing through an arbitrage node. Figure 6 displays the upper and lower bound as a function of s_0 for the BJ&N model with $p = 1$ and the modification obtained by adding different percentages of arbitrage nodes.

Figure 7 displays the upper and lower bound as a function of s_0 for the BJ&N model with $p = 3$ and the modification obtained by adding different percentages of arbitrage nodes.

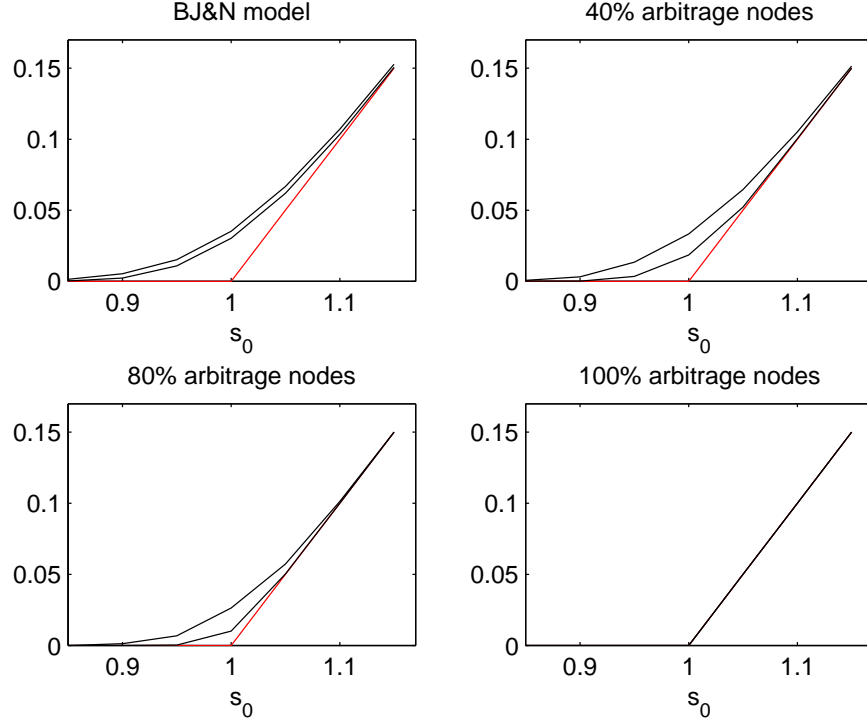


FIGURE 7. Minmax upper and lower bounds price (which are the same in this case) as a function of s_0 for $p = 3$ for the BJ&N model in the presence of arbitrage nodes in relation to the lower Merton bound (red line).

8.4. Hedging. The algorithm presented in Section 6 allow us to calculate not only the value of $\bar{V}(s_0, Z, \mathcal{M})$ but also the optimal portfolio H providing the investments along each possible trajectory in $\mathcal{S}^{\mathcal{W}}$.

On each vertex (k, j) of the data grid Γ given in Section 7.3, the dynamic upper bound $\bar{U}(k, j)$ is available and correspond to an optimal value $u(k, j) = u_{+, -}$ given by equation (7.3). Recall that $\bar{U}(k, j)$ and so $u(k, j)$, are the same for any trajectory passing trough that vertex. Then, we can define an optimal strategy $\{H_i^\uparrow\}_{i \in \mathbb{N}}$ on $S \in \mathcal{S}^{\mathcal{W}}$ defined by:

$$H_i^\uparrow(S) = u(k, j) \quad \text{if} \quad (S_i, W_i) = (s_k, w_j).$$

This optimal strategy is non-anticipative.

It is interesting to study how H^\uparrow actually approximates Z , as function of a trajectory $S \in \mathcal{S}^{\mathcal{W}}$, for an initial portfolio value X . In a short position the hedging values are given by

$$(8.3) \quad X + \sum_{i=0}^{N_{H^\uparrow}(S)-1} H_i^\uparrow(S) \Delta_i S$$

with $X \in \mathbb{R}$ the initial portfolio value.

Figure 8 shows the hedging values (8.3) with $X = \bar{V}(s_0, Z, \mathcal{M}) + 0.01$ and $X = \bar{V}(s_0, Z, \mathcal{M}) - 0.03$, for random trajectories with $s_0 = 1$ and $p = 3$ with respect to an European call Z for the general finite model obtained from Definition 14 for $\delta = \beta$ as in (8.2) for model \mathcal{M}^a . We can see that the

values from (8.3) superhedge the payoff value in the first case. For the case $V = \bar{V}(s_0, Z, \mathcal{M}) - 0.03$, the values tightly approximate the payoff values.

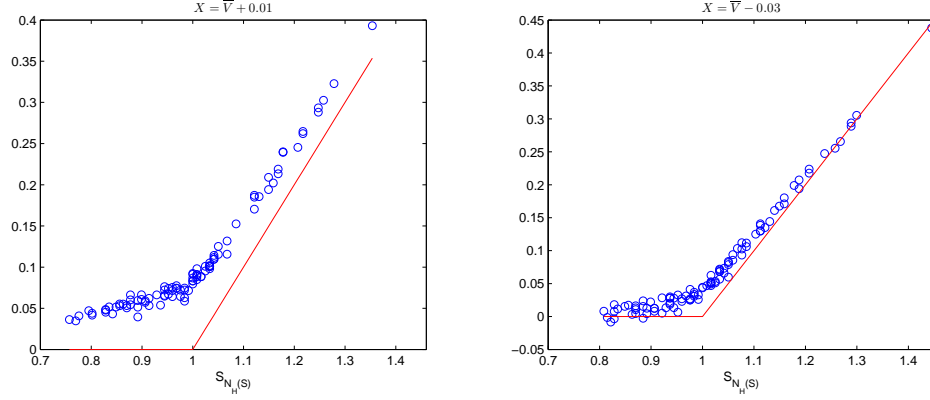


FIGURE 8. Comparison between the hedging values for $X = \bar{V}(s_0, Z, \mathcal{M}) + 0.01$ and $X = \bar{V}(s_0, Z, \mathcal{M}) - 0.03$ and the payoff values.

In a long position, the hedging values are given by

$$(8.4) \quad X - \sum_{i=0}^{N_{H^\downarrow}(S)-1} H_i^\downarrow(S) \Delta_i S$$

with $X \in \mathbb{R}$ the initial portfolio value. The underhedging portfolio H^\downarrow is computed in a similar way than the upperhedging one H^\uparrow , but using the values which gives the lower bounds $\underline{U}(k, j)$ instead of the upper bounds. Figure 9 displays the values from equation (8.4) with $X = \underline{V}(s_0, Z, \mathcal{M}) - 0.01$ and $X = \underline{V}(s_0, Z, \mathcal{M}) + 0.03$, for random trajectories with respect an European call Z . In this case, we can see that the values from (8.4) underhedge the payoff value for $X = \underline{V}(s_0, Z, \mathcal{M}) - 0.01$. For $X = \underline{V}(s_0, Z, \mathcal{M}) + 0.03$, the values better approximate the payoff values.

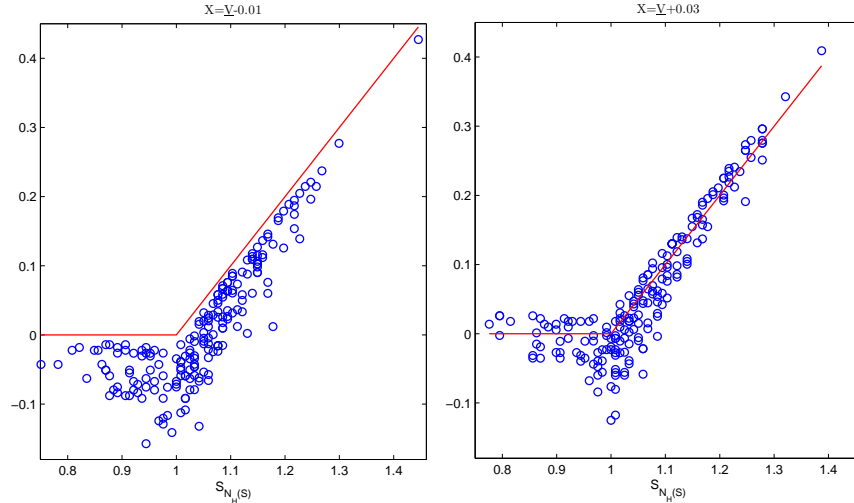


FIGURE 9. Comparison between the hedging values for $X = \underline{V}(s_0, Z, \mathcal{M}) - 0.01$ and $X = \underline{V}(s_0, Z, \mathcal{M}) + 0.03$ and the payoff values.

Finally, it is of interest to superimpose the upperhedging and lower hedging using $X = \overline{V}(s_0, Z)$ and $X = \underline{V}(s_0, Z)$ respectively. Figure 10 does this for \mathcal{M}^a and Figure 11 does it for \mathcal{M}^b . In both instances $N_2 = 100$ and $p = 3$ were chosen and $A = 2$ for the \mathcal{M}^b model.

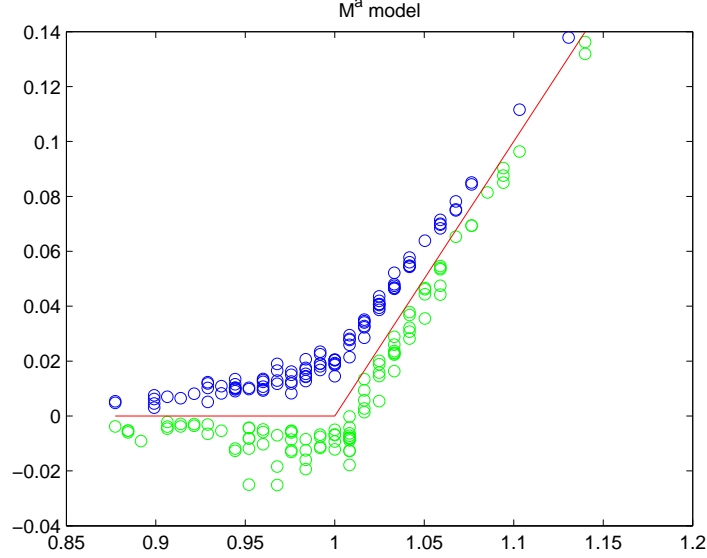


FIGURE 10. Superhedging and underhedging comparison between the hedging values for $X = \underline{V}(s_0, Z, \mathcal{M}^a)$ and $X = \overline{V}(s_0, Z, \mathcal{M}^a)$ and the payoff values.

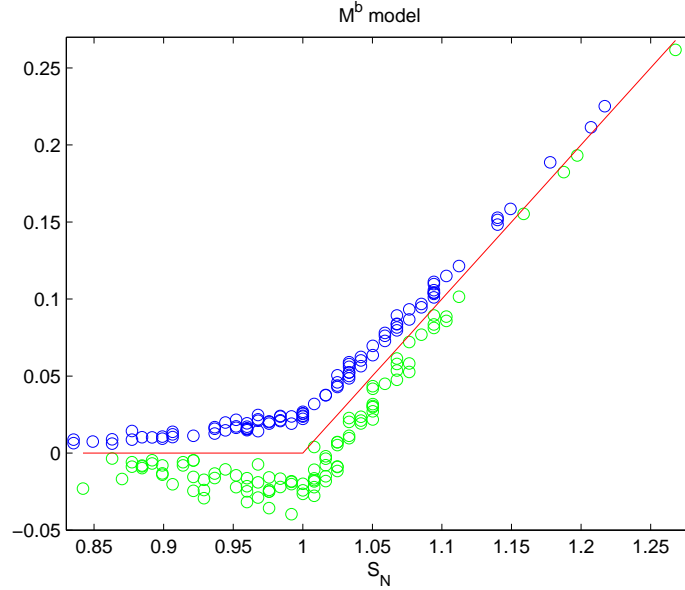


FIGURE 11. Superhedging and underhedging comparison between the hedging values for $X = \underline{V}(s_0, Z, \mathcal{M}^b)$ and $X = \overline{V}(s_0, Z, \mathcal{M}^b)$ and the payoff values.

8.5. Effect of Variable Volatility. This section is devoted to compare the behavior of the minmax bounds for different realizations of the market \mathcal{M}^b concerning to the selection of the set Q . Recall

that Q represents the possible values of quadratic variation of the trajectories in the market. First we consider markets where Q is a singleton set $\{v_j\}$ with $1 \leq j \leq m$ and $v_j < v_{j+1}$ for $1 \leq j < m$, so the corresponding markets are denoted by \mathcal{M}_j^b , $1 \leq j \leq m$. Figures 12 and 13 shows the lower and upper bound as function of increasing values of the quadratic variation for two different options and two different values of s_0 . In both figures, the bounds are computed for a call option with strike $K = 1$ and a butterfly call with strikes $K_1 = 1, K_2 = 1.1$ defined by

$$Z^f(X) = \begin{cases} (X - K_1)^+ & \text{if } X \leq \frac{K_1 + K_2}{2} \\ (K_2 - X)^+ & \text{if } X > \frac{K_1 + K_2}{2} \end{cases}.$$

The general parameters used for the models \mathcal{M}_j^b involved in those figures are given as function of v_j for $1 \leq j \leq 8$, as follows:

$$\delta_0 = \sqrt{0.0067/200}; A = 2; v_j = 25j\delta_0^2, N_1^j = N_2^j = \frac{v_j}{\delta_0^2} = 25j \text{ and } p = 3,$$

which, in turn, determine the values of the remaining parameters.

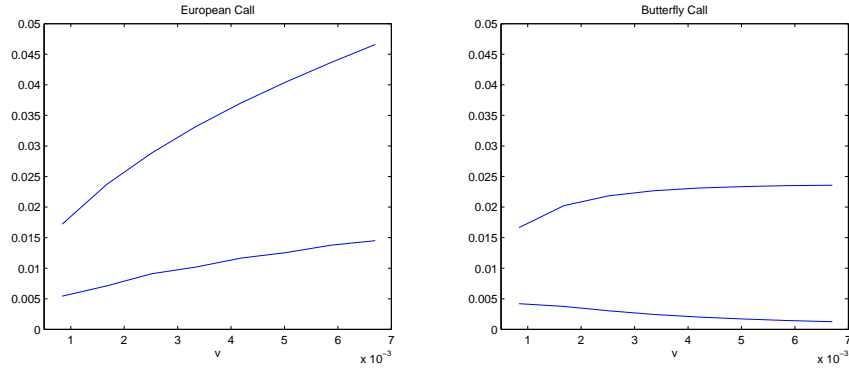


FIGURE 12. Minmax upper and lower bounds price as a function of v_j for $p = 3$ in the \mathcal{M}_j^b models with $s_0 = 1$.

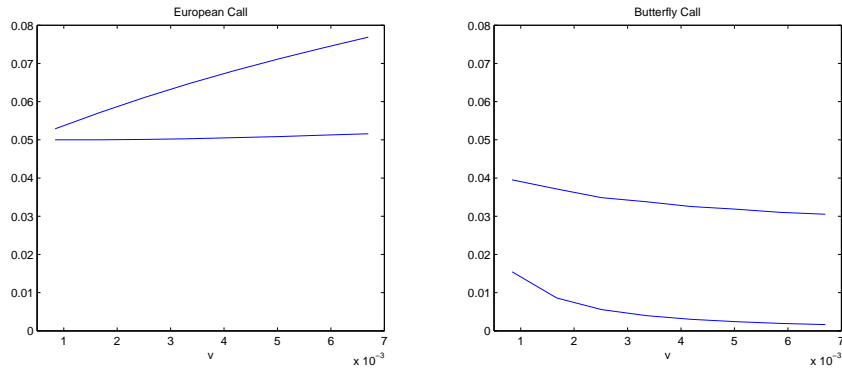


FIGURE 13. Minmax upper and lower bounds price as a function of v_j for $p = 3$ in the \mathcal{M}_j^b models with $s_0 = 1.05$.

It is observed that the bounds increases or decreases monotonically with the quadratic variation. Notice also that the payoff of the butterfly call is neither convex nor concave, we have taken $X =$

$1 = s_0$ in Figures 12 and 14 and $X = 1.05 = s_0$ in Figures 13 and 15. The payoff of the ordinary call is convex.

Next we consider market models $\tilde{\mathcal{M}}_j^b$ consisting of trajectories that have different quadratic variations belonging to $Q_j = \{v_1, \dots, v_j\}$, $1 \leq j \leq 8$. Figure 14 shows the lower and upper minmax bounds for the same european call and a butterfly call, as before, in the markets $\tilde{\mathcal{M}}_j^b$ with $s_0 = 1$, as function of j . Figure 15 is similar but with $s_0 = 1.05$. Comparing with Figures 12 and 13, it can be seen that the bounds for the models $\tilde{\mathcal{M}}_j^b$ with multiple quadratic variations Q_j coincide with the ones in the models with a single quadratic variation v_j corresponding to the largest value in Q_j .

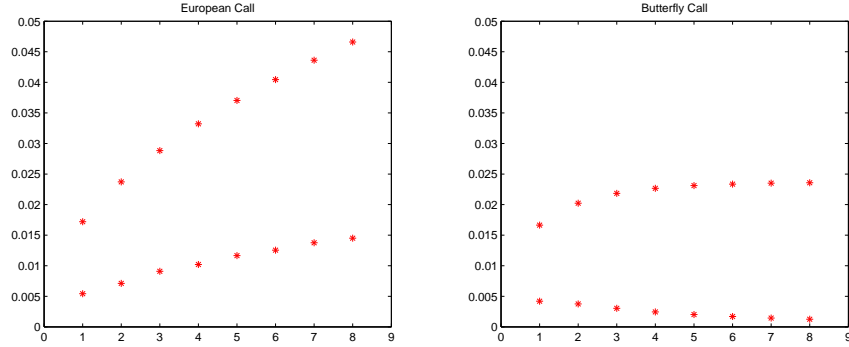


FIGURE 14. Minmax upper and lower bounds price as a function of j for $p = 3$ in the $\tilde{\mathcal{M}}_j^b$ models with $s_0 = 1$.

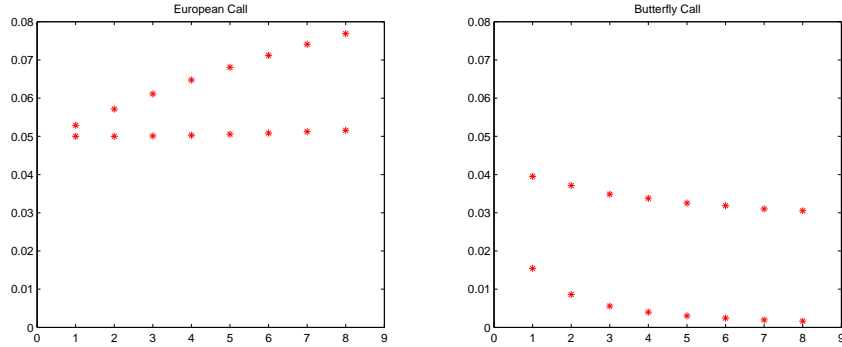


FIGURE 15. Minmax upper and lower bounds price as a function of j for $p = 3$ in the $\tilde{\mathcal{M}}_j^b$ models with $s_0 = 1.05$.

9. CONCLUSION

General results are obtained to evaluate minmax bounds in an effective way and for general classes of trajectory markets. The setting covers the case of trajectories with different volatilities. The numerical experiments indicate some of the phenomena that may occur in a trajectory based approach for the examples introduced. The design of trajectory sets can be tailored to different needs in particular by taking stopping times samples of continuous time martingale processes one can see that the risk neutral price for attainable options is a special case of trajectory based minmax

price (see [9]). Designing suitable trajectory sets for different setups seem to be a relevant task; we have provided an example dealing with quadratic variation which could be expanded in several ways, we expect to do so in a future publication.

APPENDIX A. MINMAX FUNCTIONS RESULTS

This Appendix provides the main results on minmax function and the relation with the boundedness of \bar{V} and \underline{V} .

Definition 15. *Given a finite sequence of stopping times $(v_i)_{i=1}^m$ with $v_i < v_{i+1}$ for $1 \leq i < m$, and a real sequence $(a_i)_{i=1}^m$, set (set $v_0 = 0$ for convenience),*

$$(A.1) \quad A_l(S) = \sum_{j=i}^m a_j \quad \text{if } v_{i-1}(S) \leq l < v_i(S) \quad \text{for } 1 \leq i \leq m, \quad \text{and } A_l(S) = 0 \quad \text{for } l \geq v_m(S).$$

Also, for $H \in \mathcal{H}$, define the functions $H_i^{(A)} : \mathcal{S} \rightarrow \mathbb{R}$, for $S \in \mathcal{S}$, by

$$H_i^{(A)}(S) = H_i(S) + A_i(S) \quad \text{if } 0 \leq i < v_m(S) \quad \text{and} \quad H_i^{(A)}(S) = H_i(S) \quad \text{if } i \geq v_m(S).$$

The fact that $H^{(A)} = (H_i^{(A)})_{i \geq 0}$, in the above definition, is a portfolio on \mathcal{S} , for any $H \in \mathcal{H}$, is proven in Lemma 2, Appendix A.

Proposition 3 (Proof at the end of this Appendix). *Let $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ be a discrete market and Z a function defined on \mathcal{S} . Consider a finite sequence of stopping times $(v_i)_{i=1}^m$ with $v_i < v_{i+1}$ for $1 \leq i < n$, a real sequence $(a_i)_{i=1}^m$, and $b \in \mathbb{R}$. Fix $S^* \in \mathcal{S}$ and an integer $k \geq 0$. Then the following statements are valid:*

- (1) *If Z is an upper minmax function and $0^{(A^1)} \in \mathcal{H}$, then:*

$$\bar{V}_k(S^*, Z, \mathcal{M}) \leq A_0^1 s_0 + B^1.$$

- (2) *If Z is a lower minmax function and $0^{(-A^2)} \in \mathcal{H}$, then:*

$$\underline{V}_k(S^*, Z, \mathcal{M}) \geq A_0^2 s_0 + B^2.$$

Furthermore:

- (3) *If Z is a lower minmax function and either of the two statements below hold:*
 (a) *\mathcal{M} is conditionally 0-neutral at S^* and k and for any $H \in \mathcal{H}$, $H^{(-A^3)} \in \mathcal{H}$.*
 (b) *\mathcal{M} is n -bounded such that \mathcal{S} satisfies the local 0-neutral property and v_m is bounded.*

Then:

$$(A.2) \quad \bar{V}_k(S^*, Z, \mathcal{M}) \geq A_0^3 s_0 + B^3.$$

- (4) *If Z is an upper minmax function and either of the two statements below hold:*
 (a) *\mathcal{M} is conditionally 0-neutral at S^* and k and for any $H \in \mathcal{H}$, $H^{(A^4)} \in \mathcal{H}$*
 (b) *\mathcal{M} is n -bounded such that \mathcal{S} satisfies the local 0-neutral property and v_m is bounded.*

Then:

$$\underline{V}_k(S^*, Z, \mathcal{M}) \leq A_0^4 s_0 + B^4.$$

Where for $1 \leq i \leq 4$ the sequences $(A_l^i)_{l \geq 0}$ are given by (A.1), and $B^i = \sum_{l=0}^{k-1} A_l^i(S^*) \Delta_l S^* + b$ respectively, for each item.

Observe first that, for any finite sequence of stopping times $(v_i)_{i=1}^m$ with $v_i < v_{i+1}$ for $1 \leq i < m$, and a real sequence $(a_i)_{i=1}^m$ and $S \in \mathcal{S}$: $a_i S_{v_i(S)} = a_i S_0 + \sum_{j=0}^{v_i(S)-1} a_i \Delta_j S$. Then

$$(A.3) \quad \sum_{i=1}^m a_i S_{v_i(S)} + b = \sum_{l=0}^{v_m(S)-1} A_l(S) \Delta_l S + A_0 S_0 + b.$$

The next Lemma prove that the sequence defined on Definition 15 is a portfolio on \mathcal{S} .

Lemma 2. For $H \in \mathcal{H}$, the sequence $H^{(A)} = (H_i^{(A)})_{i \geq 0}$ is non-anticipative. Consequently defining

$$V_{H^{(A)}}(0, S_0) = A_0 \quad \text{and} \quad N_{H^{(A)}}(S) = \max\{N_H(S), v_n(S)\},$$

$H^{(A)}$ can be consider as a portfolio on \mathcal{S} .

Proof. It is enough to prove that the functions A_l , defined by (A.1) for $0 \leq l < v_m$, are non anticipative. Hence, assume for $S, S' \in \mathcal{S}$, that $S_j = S'_j$ for $0 \leq j \leq l$. It follows from (A.1) that there exists $1 \leq i_0 \leq n$ such that

$$(A.4) \quad A_l(S) = \sum_{j=i_0}^m a_j, \quad \text{with} \quad v_{i_0-1}(S) \leq l < v_{i_0}(S).$$

By hypothesis $S_j = S'_j$ for $0 \leq j \leq v_{i_0-1}(S)$, then $v_{i_0-1}(S) = v_{i_0-1}(S')$. Also it must be $l < v_{i_0}(S')$, if not $l \geq v_{i_0}(S') = v_{i_0}(S)$ in contradiction with (A.4). Thus $A_l(S) = A_l(S')$. \square

Corollary 4. Fix $S^* \in \mathcal{S}$, then for any $S \in \mathcal{S}_{(S^*, k)}$

$$\sum_{l=0}^{k-1} A_l(S) \Delta_l S = \sum_{l=0}^{k-1} A_l(S^*) \Delta_l S^* \equiv A_k^*.$$

Next natural Proposition gives the key statements for the boudness of $\bar{V}(Z)$ and $\underline{V}(Z)$.

Proposition 4. Let $S^* \in \mathcal{S}$ be fixed, and $k \geq 0$, then

(1) $\bar{V}_k(S^*, Z, \mathcal{M}) < \infty$ if and only if there exists $b \in \mathbb{R}$ and $H^b \in \mathcal{H}$ such that

$$(A.5) \quad Z(S) \leq \sum_{i=k}^{N_{H^b}(S)-1} H_i^b(S) \Delta_i S + b, \quad \text{for all} \quad S \in \mathcal{S}_{(S^*, k)}.$$

In any case $\bar{V}_k(S^*, Z, \mathcal{M}) \leq b$.

(2) If there exists $b \in \mathbb{R}$ and $H^b \in \mathcal{H}$ such that

$$(A.6) \quad Z(S) \geq \sum_{i=k}^{N_{H^b}(S)-1} H_i^b(S) \Delta_i S + b, \quad \text{for all} \quad S \in \mathcal{S}_{(S^*, k)},$$

and either of the two statements below hold:

(a) \mathcal{M} is conditionally 0-neutral at S^* and k and for any $H \in \mathcal{H}$, \tilde{H} defined by $\tilde{H}_i = H_i$ if $i \leq k$ and $\tilde{H}_i = H_i - H_i^b$ if $i > k$, with $N_{\tilde{H}} = \max\{N_H, N_{H^b}\}$ and $V_{\tilde{H}}(s_0, 0) = V_H(s_0, 0)$, belongs to \mathcal{H} .

(b) \mathcal{M} is n -bounded such that \mathcal{S} satisfies the local 0-neutral property.

Then $\bar{V}_k(S^*, Z, \mathcal{M}) > -\infty$.

Proof. Proof of part (1). Since $\bar{V}_k(S^*, Z, \mathcal{M}) < \infty$, there exist $H^b \in \mathcal{H}$ and $b \in \mathbb{R}$ such that

$$\sup_{S \in \mathcal{S}_{(S^*, k)}} [Z(S) - \sum_{i=k}^{N_{H^b}(S)} H_i^b(S) \Delta_i S] \leq b.$$

From where (A.5) holds. Conversely, if (A.5) is valid, it is clear that

$$\bar{V}_k(S^*, Z, \mathcal{M}) \leq \sup_{S \in \mathcal{S}_{(S^*, k)}} [Z(S) - \sum_{i=k}^{N_{H^b}(S)} H_i^b(S) \Delta_i S] \leq b.$$

Proof of part (2): By the general hypothesis

$$\begin{aligned} \bar{V}_k(S^*, Z, \mathcal{M}) &\geq \inf_{H \in \mathcal{H}} \left\{ \sup_{S \in \mathcal{S}_{(S^*, k)}} \left[\sum_{i=k}^{N_H^b(S)-1} H_i^b(S) \Delta_i S - \sum_{i=k}^{N_H(S)-1} H_i(S) \Delta_i S \right] + b \right\} = \\ (A.7) \quad &= \inf_{H \in \mathcal{H}} \left\{ \sup_{S \in \mathcal{S}_{(S^*, k)}} \left[- \sum_{i=k}^{N_H(S)-1} \tilde{H}_i(S) \Delta_i S \right] \right\} + b \end{aligned}$$

Using now the conditions in (a), it follows that

$$\bar{V}_k(S^*, Z, \mathcal{M}) \geq \inf_{H \in \mathcal{H}} \left\{ \sup_{S \in \mathcal{S}_{(S^*, k)}} \left[- \sum_{i=k}^{N_H(S)-1} H_i(S) \Delta_i S \right] \right\} + b = b.$$

For the hypothesis (b), consider the set of portfolios $\tilde{\mathcal{H}}$ consisting of all \tilde{H} with $H \in \mathcal{H}$ defined in (a), then the market $\mathcal{S} \times \tilde{\mathcal{H}}$ is n -bounded and local 0-neutral, and then, by Proposition 2, conditionally 0-neutral. Thus the right hand side of (A.7) is equal to b . \square

Proposition 4 holds in a more general scenario. The n -bounded condition in the second part can be replaced by the initially bounded condition defined in [9], as follows.

Definition 16. Given a discrete market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ and $H \in \mathcal{H}$; we will call N_H initially bounded if there exists a bounded function $\rho : \mathcal{S} \rightarrow \mathbb{N}$ (which may depend on H) such that for all $S \in \mathcal{S}$:

$$(A.8) \quad N_H \text{ is bounded on } \mathcal{S}_{(S, \rho(S))}.$$

Under this hypothesis, Theorem 2 keep holding and then, we can reformulate Proposition 3. Observe that if N_H is bounded, it is initially bounded, $\rho = N_H$ satisfies the definition.

As the present work focus in n -bounded markets, we present the proof of Proposition 3 for this kind of markets.

Proof. of Proposition 3. Fix $S \in \mathcal{S}_{(S^*, k)}$. Proof of item (1). By (A.3) and Corollary 4

$$Z(S) \leq \sum_{i=0}^{k-1} A_i(S^*) \Delta_i S^* + \sum_{i=k}^{v_m(S)-1} A_i(S) \Delta_i S + A_0 s_0 + b = \sum_{i=k}^{N_{0^{(A)}}(S)-1} 0_i^{(A)}(S) \Delta_i S + A_0 s_0 + B.$$

Since $0^{(A)} \in \mathcal{H}$, Proposition 4, 1. gives

$$\bar{V}_k(S^*, Z, \mathcal{M}) \leq A_0 s_0 + B.$$

Proof of item (2). From hypothesis $-Z(S) \leq \sum_{i=1}^m -a_i S_{v_i(S)} - b$, and $0^{(-A)} \in \mathcal{H}$, it follows from (1) that

$$\underline{V}_k(S^*, Z, \mathcal{M}) = -\bar{V}_k(S^*, -Z, \mathcal{M}) \geq A_0 s_0 + B.$$

Proof of item (3). For any $H \in \mathcal{H}$ it follows from (A.3), and similar computation as in the proof of part (1), that

$$\begin{aligned}
 Z(S) - \sum_{i=k}^{N_H(S)-1} H_i(S) \Delta_i S &\geq \sum_{i=k}^{v_m(S)-1} A_i(S) \Delta_i S - \sum_{i=k}^{N_H(S)-1} H_i(S) \Delta_i S + A_0 s_0 + B = \\
 (A.9) \quad &= - \sum_{i=k}^{N_{H^{(-A)}}(S)-1} H_i^{(-A)}(S) \Delta_i S + A_0 s_0 + B.
 \end{aligned}$$

Under assumption (a) in item (3), we know that $H^{(-A)} \in \mathcal{H}$, therefore by 0-conditional property,

$$\begin{aligned}
 \sup_{S \in \mathcal{S}_{(S^*, k)}} [Z(S) - \sum_{i=k}^{N_H(S)-1} H_i(S) \Delta_i S] &\geq A_0 s_0 + B + \sup_{S \in \mathcal{S}_{(S^*, k)}} [- \sum_{i=k}^{N_{H^{(-A)}}(S)-1} H_i^{(-A)}(S) \Delta_i S] \geq \\
 A_0 s_0 + B + \inf_{\tilde{H} \in \mathcal{H}} \sup_{S \in \mathcal{S}_{(S^*, k)}} [- \sum_{i=k}^{N_{\tilde{H}}(S)-1} \tilde{H}_i(S) \Delta_i S] &= A_0 s_0 + B,
 \end{aligned}$$

Assume now (b) in item (3) and $M \in \mathbb{R}$ such that $v_m(S) \leq M$ for all $S \in \mathcal{S}$. Let $\tilde{\mathcal{H}}$ be any set containing the portfolios H and $H^{(-A)}$ for each $H \in \mathcal{H}$. Then, the market $\mathcal{S} \times \tilde{\mathcal{H}}$ is N -bounded, where $N = \max\{n, M\}$. Thus Proposition 2 shows that $\mathcal{S} \times \tilde{\mathcal{H}}$ is conditionally 0-neutral at all nodes, in particular at S^* , and k ; therefore, taking the supremum over $\mathcal{S}_{(S^*, k)}$ in both sides of (A.9), evaluating the infimum over $H \in \mathcal{H}$ in the right hand side, and using the conditional 0-neutral property of $\mathcal{S} \times \tilde{\mathcal{H}}$ we obtain (A.2).

The proof of item (4) follows from (3) in a similar way than (2) from (1). \square

APPENDIX B. SOME TECHNICAL RESULTS

Here are located some definitions and auxiliary lemmas required for results in subsections 4.1 and 4.2

B.1. auxiliary results for subsection 4.1.

Definition 17. Consider a discrete market model $\mathcal{M} = \mathcal{S} \times \mathcal{H}$, and a function Z defined on \mathcal{S} . Fix $k \geq 0$, and $S^k \in \mathcal{S}$. Set $\hat{s}_0 = S^k$. For $(S, H) \in \mathcal{M}$, define

$$\begin{aligned}
 \hat{S} &\equiv (S_{i+k})_{i \geq 0}, \quad \hat{H}_i(\hat{S}) \equiv H_{i+k}(S), \quad i \geq 0, \quad \hat{H} \equiv (\hat{H}_i)_{i \geq 0}, \quad \text{with } V_{\hat{H}}(0, \hat{s}_0) = V_H(k, S^k) \quad \text{and} \\
 N_{\hat{H}(\hat{S})} &= N_H(S) - k \text{ if } N_H(S) > k, \quad \text{and} \quad N_{\hat{H}(\hat{S})} = 1 \text{ if } N_H(S) \leq k.
 \end{aligned}$$

Also define

$$\widehat{\mathcal{S}} \equiv \{\hat{S} : S \in \mathcal{S}_{(S^k, k)}\}, \quad \widehat{\mathcal{H}} \equiv \{\hat{H} : H \in \mathcal{H}\}, \quad \widehat{\mathcal{M}}_k \equiv \widehat{\mathcal{S}} \times \widehat{\mathcal{H}},$$

and for any $\hat{S} \in \widehat{\mathcal{S}}$,

$$\hat{Z}(\hat{S}) \equiv Z(S).$$

Lemma 3. Under the conditions of Definition 17, for any $k \geq 0$ and $S^k \in \mathcal{S}$,

- (1) $\widehat{\mathcal{M}}_k \equiv \widehat{\mathcal{S}} \times \widehat{\mathcal{H}}$ is a discrete market model, with initial value $\hat{s}_0 \equiv S^k$. Moreover it is n -bounded if \mathcal{M} is $n+k$ -bounded.
- (2) Assuming \mathcal{M} is $n+k$ -bounded, for any $S \in \mathcal{S}_{(S^k, k)}$,

$$\overline{U}_i(\hat{S}, \hat{Z}, \widehat{\mathcal{M}}_k) = \overline{U}_{i+k}(S, Z, \mathcal{M}) \quad \text{for } 0 \leq i \leq n.$$

Proof. By definition, $\widehat{\mathcal{S}}$ consist of sequences in \mathbb{R}^N , with $\hat{S}_0 = S_k = S_k^k = \hat{s}_0$ for any $\hat{S} \in \widehat{\mathcal{S}}$. $\widehat{\mathcal{H}}$ is a family of sequences of functions $(\hat{H})_{i \geq 0}$ with $\hat{H}_i : \widehat{\mathcal{S}} \rightarrow \mathbb{R}$ non-anticipative, since if $\hat{S}'_j = \hat{S}_j$, for $0 \leq j \leq i$ then $S'_{j+k} = S_{j+k}$, for $0 \leq j+k \leq i+k$, thus $\hat{H}_i(\hat{S}') = H_{i+k}(S') = H_{i+k}(S) = \hat{H}_i(\hat{S})$. Moreover if $i \geq N_{\hat{H}}(\hat{S})$, then $\hat{H}_i(\hat{S}) = H_{i+k}(S) = 0$, and Therefore $\widehat{\mathcal{H}}$ is a set of portfolios on $\widehat{\mathcal{S}}$. This proves (1).

For (2), we proceed by induction backwards. For each $S \in \mathcal{S}_{(S^k, k)}$ and $H \in \mathcal{H}$, if

$$M(\hat{S}) \equiv \sup_{H \in \mathcal{H}} N_{\hat{H}}(\hat{S}) = n,$$

then $M(S) = n + k$ and $\overline{U}_n(\hat{S}, \hat{Z}, \widehat{\mathcal{M}}_k) = \hat{Z}(\hat{S}) = Z(S) = \overline{U}_{n+k}(S, Z, \mathcal{M})$. But if $M(\hat{S}) < n$, then $M(S) < n + k$ and $\overline{U}_n(\hat{S}, \hat{Z}, \widehat{\mathcal{M}}_k) = 0 = \overline{U}_{n+k}(S, Z, \mathcal{M})$.

Now assume (2) is valid for some $0 < i \leq n$. Fix any $S \in \mathcal{S}_{(S^k, k)}$, if $i - 1 \geq M(\hat{S})$, we have that $i + k - 1 \geq M(S)$ and

$$\overline{U}_{i-1}(\hat{S}, \hat{Z}, \widehat{\mathcal{M}}_k) = \hat{Z}(\hat{S}) = Z(S) = \overline{U}_{i+k-1}(S, Z, \mathcal{M}) \quad \text{or} \quad \overline{U}_{i-1}(\hat{S}, \hat{Z}, \widehat{\mathcal{M}}_k) = 0 = \overline{U}_{i+k-1}(S, Z, \mathcal{M}).$$

On the other hand when $0 \leq i - 1 < M(\hat{S})$, it follows

$$\begin{aligned} \overline{U}_{i-1}(\hat{S}, \hat{Z}, \widehat{\mathcal{M}}_k) &= \inf_{\hat{H} \in \widehat{\mathcal{H}}} \{ \sup_{\hat{S}' \in \widehat{\mathcal{S}}_{(\hat{S}, i-1)}} [\overline{U}_i(\hat{S}', \hat{Z}, \widehat{\mathcal{M}}_k) - \hat{H}_{i-1}(\hat{S}) \Delta_{i-1} \hat{S}'] \} \\ &= \inf_{H \in \mathcal{H}} \{ \sup_{S' \in \mathcal{S}_{(S, i)}} [\overline{U}_{i+k}(S', Z, \mathcal{M}) - H_{i+k-1}(S) \Delta_{i+k-1} S'] \} = \\ &= \overline{U}_{i+k-1}(S, Z, \mathcal{M}). \end{aligned}$$

□

To prove Theorem 3, we need the following results for induction procedures.

Lemma 4. *Given a n -bounded discrete market model $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ and a function Z on \mathcal{M} .*

(1) *Fix $H \in \mathcal{H}$, then*

$$\begin{aligned} \text{(B.1)} \quad \sup_{S \in \mathcal{S}} [Z(S) - \sum_{i=0}^{N_H(S)-1} H_i(S) \Delta_i S] &= \sup_{S^1 \in \mathcal{S}} [-H_0(S_0) \Delta_0 S^1 + \sup_{S^2 \in \mathcal{S}_{(S^1, 1)}} [-H_1(S^2) \Delta_1 S^2 + \dots \\ &\quad \dots + \sup_{S^n \in \mathcal{S}_{(S^{n-1}, n-1)}} [Z(S^n) - H_{n-1}(S^{n-1}) \Delta_{n-1} S^n] \dots]]. \end{aligned}$$

(2) *Furthermore,*

$$\begin{aligned} \overline{U}_0(S_0, Z, \mathcal{M}) &= \inf_{H^0 \in \mathcal{H}} \sup_{S^1 \in \mathcal{S}^n} \{ -H_0^0(S^1) \Delta_0 S^1 + \inf_{H^1 \in \mathcal{H}} \sup_{S^2 \in \mathcal{S}_{(S^1, 1)}} \{ -H_1^1(S^1) \Delta_1 S^2 + \dots \\ &\quad \dots + \inf_{H^{n-1} \in \mathcal{H}} \sup_{S^n \in \mathcal{S}_{(S^{n-1}, n-1)}} [Z(S^n) - H_{n-1}^{n-1}(S^{n-1}) \Delta_{n-1} S^n] \dots \} \}. \end{aligned}$$

Proof. Since (1) holds for $n = 1$, we proceed by induction. Assume then that the equality holds for n -bounded market models, and consider $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ an $(n+1)$ -bounded market. Fixed $S^1 \in \mathcal{S}$,

by inductive hypothesis and Lemma 3, applied for $k = 1$, we obtain

$$\begin{aligned}
\sup_{S \in \mathcal{S}_{(S^1, 1)}} [Z(S) - \sum_{i=1}^{N_H(S)-1} H_i(S) \Delta_i S] &= \sup_{\widehat{S} \in \widehat{\mathcal{S}}} [Z(\widehat{S}) - \sum_{i=0}^{N_{\widehat{H}}(\widehat{S})-1} \widehat{H}_i(\widehat{S}) \Delta_i \widehat{S}] = \\
&= \sup_{\widehat{S}^2 \in \widehat{\mathcal{S}}} [-\widehat{H}_0(\widehat{S}_0^2) \Delta_0 \widehat{S}^2 + \dots + \sup_{\widehat{S}^n \in \widehat{\mathcal{S}}_{(\widehat{S}^{n-1}, n-1)}} [Z(\widehat{S}^n) - \widehat{H}_{n-1}(\widehat{S}^{n-1}) \Delta_{n-1} \widehat{S}^n] \dots] = \\
&= \sup_{S^2 \in \mathcal{S}_{(S^1, 1)}} [-H_1(S^1) \Delta_1 S^2 + \dots + \sup_{S^{n+1} \in \mathcal{S}_{(S^n, n)}} [Z(S^{n+1}) - H_n(S^n) \Delta_n S^{n+1}] \dots].
\end{aligned}$$

It follows that

$$\begin{aligned}
Z(S^1) - \sum_{i=0}^{N_H(S^1)-1} H_i(S^1) \Delta_i S^1 &\leq -H_0(S^1) \Delta_0 S^1 + \sup_{S \in \mathcal{S}_{(S^1, 1)}} [Z(S) \pm \sum_{i=1}^{N_H(S)-1} H_i(S) \Delta_i S] = \\
&= -H_0(S^1) \Delta_0 S^1 + \sup_{S^2 \in \mathcal{S}_{(S^1, 1)}} [-H_1(S^1) \Delta_1 S^2 + \dots + \sup_{S^{n+1} \in \mathcal{S}_{(S^n, n)}} [Z(S^{n+1}) - H_n(S^n) \Delta_n S^{n+1}] \dots],
\end{aligned}$$

consequently

$$\begin{aligned}
\sup_{S \in \mathcal{S}} [Z(S) - \sum_{i=0}^{N_H(S)-1} H_i(S) \Delta_i S] &\leq \sup_{S^1 \in \mathcal{S}} [-H_0(S_0) \Delta_0 S^1 + \sup_{S^2 \in \mathcal{S}_{(S^1, 1)}} [-H_1(S^2) \Delta_1 S^2 + \dots \\
&\quad \dots + \sup_{S^n \in \mathcal{S}_{(S^{n-1}, n-1)}} [Z(S^n) - H_{n-1}(S^{n-1}) \Delta_{n-1} S^n] \dots].
\end{aligned}$$

To obtain the reverse inequality, it is enough to assume that $\sup_{S \in \mathcal{S}} [Z(S) - \sum_{i=0}^{N_H(S)-1} H_i(S) \Delta_i S] < \infty$, consequently

$$\sup_{S \in \mathcal{S}_{(S^1, 1)}} [Z(S) - \sum_{i=1}^{N_H(S)-1} H_i(S) \Delta_i S] < \infty,$$

if this were not so we would have that for any $M \in \mathbb{N}$ there exists $S^* \in \mathcal{S}_{(S^1, 1)}$ such that

$$M < -H_0(S^1) \Delta_0 S^1 + Z(S^*) - \sum_{i=1}^{N_H(S^*)-1} H_i(S^*) \Delta_i S^* = Z(S^*) - \sum_{i=0}^{N_H(S^*)-1} H_i(S^*) \Delta_i S^*.$$

Let $m \in \mathbb{N}$, then there exists $S^m \in \mathcal{S}_{(S^1, 1)}$ such that

$$\begin{aligned}
&-H_0(S^1) \Delta_0 S^1 + \sup_{S \in \mathcal{S}_{(S^1, 1)}} [Z(S) - \sum_{i=1}^{N_H(S)-1} H_i(S) \Delta_i S] < \\
&< \frac{1}{m} - H_0(S^1) \Delta_0 S^1 + Z(S^m) - \sum_{i=1}^{N_H(S^m)-1} H_i(S^m) \Delta_i S^m = \frac{1}{m} + Z(S^m) - \sum_{i=0}^{N_H(S^m)-1} H_i(S^m) \Delta_i S^m.
\end{aligned}$$

Thus

$$\begin{aligned}
\sup_{S \in \mathcal{S}} [Z(S) - \sum_{i=0}^{N_H(S)-1} H_i(S) \Delta_i S] &\leq \sup_{S \in \mathcal{S}} [-H_0(S^1) \Delta_0 S^1 + \sup_{S \in \mathcal{S}_{(S^1, 1)}} [Z(S) - \sum_{i=1}^{N_H(S)-1} H_i(S) \Delta_i S]] \\
&\leq \frac{1}{m} + \sup_{S \in \mathcal{S}} [Z(S) - \sum_{i=0}^{N_H(S)-1} H_i(S) \Delta_i S].
\end{aligned}$$

Since m is arbitrary, the equality (B.1) follows.

(2) will be proven by induction too, since for $n = 1$,

$$\bar{U}_0(S_0, Z, \mathcal{M}^1) = \inf_{H^0 \in \mathcal{H}} \sup_{S^1 \in \mathcal{S}^1} [Z(S^1) - H_0^0(S^1) \Delta_0 S^1].$$

Assume now that the equality holds for n -bounded markets, and consider $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ to be $(n+1)$ -bounded. For a fixed $S^1 \in \mathcal{S}$, from the inductive hypothesis and Lemma 3, it follows that

$$\begin{aligned} \bar{U}_1(S^1, Z, \mathcal{M}^{n+1}) &= \bar{U}_0(\hat{S}_0, \hat{Z}, \widehat{\mathcal{M}}_1^{n+1}) = \\ &= \inf_{\hat{H}^1 \in \widehat{\mathcal{H}}} \sup_{\hat{S}^2 \in \widehat{\mathcal{S}}} \{-\hat{H}_0^1(\hat{S}^2) \Delta_0 \hat{S}^2 + \dots + \inf_{\hat{H}^n \in \widehat{\mathcal{H}}} \sup_{\hat{S}^{n+1} \in \widehat{\mathcal{S}}_{(\hat{S}^1, n)}} [Z(\hat{S}^{n+1}) - \hat{H}_{n-1}^n(\hat{S}^n) \Delta_{n-1} \hat{S}^{n+1}]\} = \\ &= \inf_{H^1 \in \mathcal{H}} \sup_{S^2 \in \mathcal{S}_{(S^1, 1)}} [-H_1^1(S^1) \Delta_1 S^2 + \dots + \inf_{H^n \in \mathcal{H}} \sup_{S^{n+1} \in \mathcal{S}_{(S^n, n)}} [Z(S^{n+1}) - H_n^n(S^n) \Delta_n S^{n+1}]]. \end{aligned}$$

From which the desired result is clear. \square

Lemma 5. Consider the n -bounded market $\widehat{\mathcal{M}}_1 \equiv \widehat{\mathcal{S}} \times \widehat{\mathcal{H}}$, given in definition 17, for $k = 1$ and some $S^1 \in \mathcal{S}$ in an $(n+1)$ -bounded market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$. Then $\widehat{\mathcal{H}}$ is FULL if so is also \mathcal{H} .

Proof. Assume \mathcal{H} is FULL. Let $1 \leq k \leq n-1$, $\hat{H}' \in \widehat{\mathcal{H}}$, and $\hat{S}' \in \widehat{\mathcal{S}}$. We are going to prove that for $k \leq j \leq n-1$, any function

$$h : \widehat{\mathcal{S}}_{(\hat{S}', k)} \rightarrow I_{\widehat{\mathcal{S}}_{(\hat{S}', k)}}^j$$

non-anticipative with respect to j , is the j -coordinate of a portfolio $\hat{H} \in \widehat{\mathcal{H}}$. For it, we will find $H \in \mathcal{H}$ such that $\hat{H}_j = h$ on $\widehat{\mathcal{S}}_{(\hat{S}', k)}$.

We need to show that $\hat{S} \in \widehat{\mathcal{S}}_{(\hat{S}', k)}$ if and only if $S \in \mathcal{S}_{(S', k+1)}$. Let $0 \leq i \leq k$, and $\hat{S} \in \widehat{\mathcal{S}}_{(\hat{S}', k)}$, then

$$S_{i+1} = \hat{S}_i = \hat{S}'_i = S'_{i+1}. \text{ On the other hand } \hat{S}_i = S_{i+1} = S'_{i+1} = \hat{S}'_i.$$

If $\hat{H} \in \widehat{\mathcal{H}}$, and $\hat{S} \in \widehat{\mathcal{S}}$, $\hat{H}_j(\hat{S}) = H_{j+1}(S)$, it means that $I_{\widehat{\mathcal{S}}_{(\hat{S}', k)}}^j \subset I_{\mathcal{S}_{(S', k+1)}}^j$. Since \mathcal{H} is FULL it then follows that exists $H \in \mathcal{H}$ such that $H_{j+1} : \mathcal{S}_{(S', k+1)} \rightarrow I_{\mathcal{S}_{(S', k+1)}}^j$ is given by $H_{j+1}(S) \equiv h(\hat{S})$. \square

B.2. proofs of u-complete markets section. Consider a discrete $(n+1)$ -bounded market $\mathcal{M} = \mathcal{S} \times \mathcal{H}$. For any $H \in \mathcal{H}$ define

$$\tilde{H}_i = H_i, \quad 0 \leq i \leq n; \quad \tilde{H}_i = 0, \quad i > n; \quad \tilde{H} = (\tilde{H}_i)_{i \geq 0}; \quad V_{\tilde{H}}(0, S_0) = V_H(0, S_0),$$

$$\widetilde{\mathcal{H}} \equiv \{\tilde{H} : H \in \mathcal{H}\} \quad \text{and} \quad \widetilde{\mathcal{M}} \equiv \mathcal{S} \times \widetilde{\mathcal{H}}.$$

$$\text{For } S \in \mathcal{S} \text{ and } \tilde{H} \in \widetilde{\mathcal{H}} \text{ set } N_{\tilde{H}}(S) = \begin{cases} n & \text{if } N_H(S) = n+1. \\ N_H(S) & \text{if } N_H(S) \leq n. \end{cases}$$

$\widetilde{\mathcal{M}}$ results an n -bounded discrete market. If Z is a derivative function defined on \mathcal{S} , then \tilde{Z} is defined, for any $S \in \mathcal{S}$, by

$$\tilde{Z}(S) = \begin{cases} \bar{U}_n(S, Z, \mathcal{M}) & \text{if } M(S) = n+1. \\ Z(S) & \text{if } M(S) \leq n. \end{cases}$$

Moreover

Lemma 6. Let $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ an $(n+1)$ -bounded discrete market. Then

(a) for any $0 \leq k \leq n$, and $S \in \mathcal{S}$,

$$\overline{U}_k(S, \tilde{Z}, \widetilde{\mathcal{M}}) = \overline{U}_k(S, Z, \mathcal{M}).$$

(b) If \mathcal{M} is u -complete for Z , so is $\widetilde{\mathcal{M}}$ for \tilde{Z} .

Proof. Set $\tilde{M}(S) = \max\{N_{\tilde{H}}(S) : \tilde{H} \in \widetilde{\mathcal{H}}\}$. Reasoning by induction backwards, for $k = n$, and $S \in \mathcal{S}$,

$$\overline{U}_n(S, \tilde{Z}, \widetilde{\mathcal{M}}) = 0 = \overline{U}_n(S, Z, \mathcal{M}) \quad \text{if } n > \tilde{M}(S), \text{ (since then } n > M(S) \text{ by def.) and}$$

$$\overline{U}_n(S, \tilde{Z}, \widetilde{\mathcal{M}}) = \tilde{Z}(S) = \overline{U}_n(S, Z, \mathcal{M}) \quad \text{if } \tilde{M}(S) = n.$$

Since if $M(S) = n+1$, $\tilde{Z}(S) = \overline{U}_n(S, Z, \mathcal{M})$, and if $M(S) = n$, $\tilde{Z}(S) = Z(S) = \overline{U}_n(S, Z, \mathcal{M})$. Assume (a) is valid for some $0 < k \leq n$. If $\tilde{M}(S) \leq k-1$, then $\tilde{M}(S) = M(S)$ and, $\overline{U}_{k-1}(S, \tilde{Z}, \widetilde{\mathcal{M}}) = \overline{U}_{k-1}(S, Z, \mathcal{M})$, since its common value is 0 or $\tilde{Z}(S) = Z(S)$. If $k-1 < \tilde{M}(S)$, then $k-1 < \tilde{M}(S)$ ($k-1 \geq \tilde{M}$ implies $M(S) > \tilde{M}(S)$, then $\tilde{M}(S) = n > k-1$!), and by inductive hypothesis and definition of $\widetilde{\mathcal{M}}$,

$$\begin{aligned} \overline{U}_{k-1}(S, \tilde{Z}, \widetilde{\mathcal{M}}) &= \inf_{\tilde{H} \in \widetilde{\mathcal{H}}} \sup_{S' \in \mathcal{S}_{(S, k-1)}} [\overline{U}_k(S', \tilde{Z}, \widetilde{\mathcal{M}}) - \tilde{H}_{k-1}(S') \Delta_{k-1} S'] \\ &= \inf_{H \in \mathcal{H}} \sup_{S' \in \mathcal{S}_{(S, k-1)}} [\overline{U}_k(S', Z, \mathcal{M}) - H_{k-1}(S') \Delta_{k-1} S'] = \overline{U}_{k-1}(S, Z, \mathcal{M}). \end{aligned}$$

For (b), let $S^* \in \mathcal{S}$, $1 \leq k \leq n-1$ and a derivative function Z . Since \mathcal{M} is u -complete there exists $H^* \in \mathcal{H}$, such that

$$\sup_{S \in \mathcal{S}_{(S^*, k)}} [U_{k+1}(S, \tilde{Z}, \widetilde{\mathcal{M}}) - \widetilde{H}_k^*(S) \Delta_k S] = \sup_{S \in \mathcal{S}_{(S^*, k)}} [U_{k+1}(S, Z, \mathcal{M}) - H_k^*(S) \Delta_k S] = \overline{U}_k(S^*, Z, \mathcal{M}) = \overline{U}_k(S^*, \tilde{Z}, \widetilde{\mathcal{M}}).$$

Last equalities hold for (a). □

Proof. Of **Theorem 4.** As in the proof of Theorem 3 the required equality for $n = 1$ is clear. Lets complete the proof by induction on n . Assume $\mathcal{M} = \mathcal{S} \times \mathcal{H}$ is an $(n+1)$ -bounded discrete market, which is u -complete, then by lemma 6 (b) below, $\widetilde{\mathcal{M}}$ is n -bounded and u -complete. Thus by lemma 6 (a) and inductive hypothesis,

$$\overline{U}_0(S_0, Z, \mathcal{M}) = \overline{U}_0(S_0, \tilde{Z}, \widetilde{\mathcal{M}}) = \overline{V}(S_0, \tilde{Z}, \widetilde{\mathcal{M}}).$$

By u -completeness, for any $S \in \mathcal{S}$ there exists $H^* \in \mathcal{H}$ such that

$$\overline{U}_n(S, Z, \mathcal{M}) = \sup_{S' \in \mathcal{S}_{(S, n)}} \{ \overline{U}_{n+1}(S', Z, \mathcal{M}) - H_n^*(S) \Delta_n S' \}.$$

If $M(S) = n+1$,

$$\tilde{Z}(S) = \overline{U}_n(S, Z, \mathcal{M}) = \sup_{S' \in \mathcal{S}_{(S, n)}} \{ Z(S') - H_n^*(S) \Delta_n S' \} \geq Z(S) - H_n^*(S) \Delta_n S,$$

and if $M(S) \leq n$, $\tilde{Z}(S) = Z(S) - H_n^*(S)\Delta_i S$, since $H_n^*(S) = 0$. In any case

$$\begin{aligned} \bar{V}(S_0, \tilde{Z}, \tilde{\mathcal{M}}) &= \inf_{\tilde{H} \in \tilde{\mathcal{H}}} \sup_{S \in \mathcal{S}} [\tilde{Z}(S) - \sum_{i=0}^{n-1} \tilde{H}_i(S)\Delta_i S] \geq \\ &\geq \inf_{H \in \mathcal{H}} \sup_{S \in \mathcal{S}} [Z(S) - H_n^*(S)\Delta_i S - \sum_{i=0}^{n-1} H_i(S)\Delta_i S] \geq \\ &\geq \inf_{H \in \mathcal{H}} \sup_{S \in \mathcal{S}} [Z(S) - \sum_{i=0}^n H_i(S)\Delta_i S] = \bar{V}(S_0, Z, \mathcal{M}). \end{aligned}$$

The other inequality follows from Proposition 1. \square

Proof. of **Lemma 1.** Define $G : \mathbb{R} \rightarrow \mathbb{R}$, by

$$G(u) = \sup_{S \in \mathcal{S}_{(S^*, k)}} \{\bar{U}_{k+1}(S, Z, \mathcal{M}) - u \Delta_k S\},$$

assuming that $\bar{U}_{k+1}(S, Z, \mathcal{M}) < \infty$. Since for any $S \in \mathcal{S}_{(S^*, k)}$, the functions given by $G_S(u) = \bar{U}_{k+1}(S, Z, \mathcal{M}) - u \Delta_k S$ are affine, then its supremum G is lower semicontinuous, and convex.

If $I_{S^*}^k$ is compact, by lower semicontinuity, there exists $u^* \in I_{S^*}^k$ verifying $G(u^*) = \inf_{u \in I_{S^*}^k} G(u)$.

If $I_S^k = \mathbb{R}$ and \mathcal{S} satisfies the local up-down property at S^* and k , G is also coercive. Indeed, there exist $S^+, S^- \in \mathcal{S}_{(S^*, k)}$ such that $S_{k+1}^+ - S_k = r^+ > 0$ and $S_{k+1}^- - S_k = r^- < 0$. Let $m \in \mathbb{N}$ and

$$K = \max \left\{ \left| \frac{m - \bar{U}_{k+1}(S^+, Z, \mathcal{M})}{r^+} \right|, \left| \frac{\bar{U}_{k+1}(S^-, Z, \mathcal{M}) - m}{r^-} \right| \right\}.$$

If $u > K$, $u = |u| > \frac{\bar{U}_{k+1}(S^+, Z, \mathcal{M}) - m}{r^-}$, then $m < \bar{U}_{k+1}(S^-, Z, \mathcal{M}) - u \Delta_k S^- \leq G(u)$. On the other hand, if $u < -K$, since $-u = |u| > \frac{m - \bar{U}_{k+1}(S^+, Z, \mathcal{M})}{r^+}$, then $G(u) \geq \bar{U}_{k+1}(S^+, Z, \mathcal{M}) - u \Delta_k S^+ > m$. Thus, by Corollary 4.3 in [3], from [11, Thm 7.3.1] G attains a minimizer.

Finally, by coercivity, there exists $R > 0$ such that, $G(u) > |G(0)| \geq G(0)$ if $|u| > R$. Then

$$\inf\{G(u) : |u| \leq R\} \leq G(0) \leq \inf\{G(u) : |u| > R\}.$$

\square

APPENDIX C. PROOF OF CONVEX HULL THEOREM

Proof. of **Theorem 6.** It is enough to consider that $L_i(S, Z, \mathcal{M}) < \infty$. We just prove $\bar{U}_i(S, Z, \mathcal{M}) \leq L_i(S, Z, \mathcal{M})$ because the other inequality follows immediately from Proposition 2. Let $\delta > 0$ there exist $S^\bullet \in \mathcal{S}_{(S, i)}^+$ and $S^\circ \in \mathcal{S}_{(S, i)}^-$ such that

$$L_i(S, Z, \mathcal{M}) \leq \bar{U}_{i+1}(S^\bullet, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^\bullet - S_i) + \delta.$$

Observe that, in case (1) it holds for S^\bullet and S° in the hypothesis. Also, since

$$\bar{U}_i(S, Z, \mathcal{M}) \leq \sup_{S' \in \mathcal{S}_{(S, i)}} \{\bar{U}_{i+1}(S', Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}' - S_i)\},$$

there exists $S^* \in \mathcal{S}_{(S, i)}$ such that

$$\bar{U}_i(S, Z, \mathcal{M}) \leq \bar{U}_{i+1}(S^*, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^* - S_i) + \delta.$$

Consider case (1) first. If $S_{i+1}^* \leq S_i$, it must be

$$\bar{U}_{i+1}(S^*, Z, \mathcal{M}) \leq \bar{U}_{i+1}(S^\bullet, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^\bullet - S_{i+1}^*).$$

If not

$$-(\overline{U}_{i+1}(S^\bullet, Z, \mathcal{M}) - \overline{U}_{i+1}(S^*, Z, \mathcal{M})) > -u_{(S^\bullet, S^*)}(S_{i+1}^\bullet - S_{i+1}^*) \text{ so } -u_{(S^\bullet, S^*)} > -u_{(S^\bullet, S^\circ)},$$

which leads to the contradiction

$$\overline{U}_{i+1}(S^\bullet, Z, \mathcal{M}) - u_{(S^\bullet, S^*)}(S_{i+1}^\bullet - S_i) > L_i(S, Z, \mathcal{M}).$$

Whence,

$$\overline{U}_i(S, Z, \mathcal{M}) - \delta \leq \overline{U}_{i+1}(S^\bullet, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^\bullet - S_{i+1}^*) - u_{(S^\bullet, S^\circ)}(S_{i+1}^* - S_i) = L_i(S, Z, \mathcal{M}).$$

While if $S_{i+1}^* > S_i$, in a similar way results

$$\overline{U}_{i+1}(S^*, Z, \mathcal{M}) \leq \overline{U}_{i+1}(S^\circ, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^\circ - S_{i+1}^*) - u_{(S^\bullet, S^\circ)}(S_{i+1}^* - S_i).$$

Thus

$$\begin{aligned} \overline{U}_i(S, Z, \mathcal{M}) - \delta &\leq \overline{U}_{i+1}(S^\circ, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^\circ - S_{i+1}^*) - u_{(S^\bullet, S^\circ)}(S_{i+1}^* - S_i) = \\ &= \overline{U}_{i+1}(S^\circ, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^\circ - S_i) = L_i(S, Z, \mathcal{M}). \end{aligned}$$

Then the proof for case (1) is done.

For case (2), assume first that $S_{i+1}^* \leq S_i$ and define $r = \delta \frac{S_{i+1}^\bullet - S_{i+1}^*}{S_{i+1}^\bullet - S_i} > 0$. We are going to show, by contradiction, that

$$(C.1) \quad \overline{U}_{i+1}(S^*, Z, \mathcal{M}) \leq \overline{U}_{i+1}(S^\bullet, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^\bullet - S_{i+1}^*) + r.$$

Assume

$$\overline{U}_{i+1}(S^*, Z, \mathcal{M}) > \overline{U}_{i+1}(S^\bullet, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^\bullet - S_{i+1}^*) + r.$$

then

$$-(\overline{U}_{i+1}(S^\bullet, Z, \mathcal{M}) - \overline{U}_{i+1}(S^*, Z, \mathcal{M})) > -u_{(S^\bullet, S^\circ)}(S_{i+1}^\bullet - S_{i+1}^*) + r$$

so

$$-u_{(S^\bullet, S^*)} > -u_{(S^\bullet, S^\circ)} + \frac{r}{S_{i+1}^\bullet - S_{i+1}^*}$$

which leads to

$$\begin{aligned} \overline{U}_{i+1}(S^\bullet, Z, \mathcal{M}) - u_{(S^\bullet, S^*)}(S_{i+1}^\bullet - S_i) &> \overline{U}_{i+1}(S^\bullet, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^\bullet - S_i) + r \frac{S_{i+1}^\bullet - S_i}{S_{i+1}^\bullet - S_{i+1}^*} \\ &\geq L_i(S, Z, \mathcal{M}), \end{aligned}$$

A contradiction to the definition of L_i . Then since (C.1) holds,

$$\overline{U}_i(S, Z, \mathcal{M}) - \delta \leq \overline{U}_{i+1}(S^\bullet, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^\bullet - S_{i+1}^*) - u_{(S^\bullet, S^\circ)}(S_{i+1}^* - S_i) + r,$$

now, since $r \leq \frac{2b}{a}$, it follows

$$\overline{U}_i(S, Z, \mathcal{M}) \leq L_i(S, Z, \mathcal{M}) + \delta + \rho \leq L_i(S, Z, \mathcal{M}) + \left(1 + \frac{2b}{a}\right) \delta.$$

While if $S_{i+1}^* > S_i$, in a similar way results

$$\overline{U}_{i+1}(S^*, Z, \mathcal{M}) \leq \overline{U}_{i+1}(S^\circ, Z, \mathcal{M}) - u_{(S^\bullet, S^\circ)}(S_{i+1}^\circ - S_{i+1}^*) + r$$

for $r = \delta \frac{S_{i+1}^* - S_{i+1}^o}{S_{i+1}^o - S_i^o} > 0$. Since $r \leq \frac{2b}{a}$, it follows from (C.2)

$$\bar{U}_i(S, Z, \mathcal{M}) \leq L_i(S, Z, \mathcal{M}) + \delta + \rho \leq L_i(S, Z, \mathcal{M}) + \left(1 + \frac{2d}{c}\right) \delta.$$

Then the proof for case (2) is complete. \square

APPENDIX D. AUXILIARY RESULTS

The next geometric Lemma is used in section 5.

Lemma 7. *Let $A, B, C, D, s_1, s_2, s \in \mathbb{R}$, with $s_1 < s_2$ and $s_1 \leq s \leq s_2$. If $A > B$ and $C > D$, then*

$$(D.1) \quad B - \left(\frac{B-D}{s_2-s_1}\right)(s_2-s) \leq A - \left(\frac{A-C}{s_2-s_1}\right)(s_2-s)$$

Proof. Let

$$\lambda = \frac{s_2-s}{s_2-s_1}$$

Since $s_1 \leq s \leq s_2$, it follows that $0 \leq \lambda \leq 1$. Then

$$\lambda(A-B-(C-D)) \leq A-B$$

Rearranging last inequality we have (D.1). \square

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